Adaptive Stabilization of Nonlinear Systems

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Abstract. An overview of the various parametric approaches which can be adopted to solve the problem of adaptive stabilization of nonlinear systems is presented. The Lyapunov design and two estimation designs –equation error filtering and regressor filtering– are revisited. This allows us to unify and generalize most of the available results on the topic and to propose a classification depending on the required extra assumptions – matching conditions or growth conditions.

1 Problem Statement and Assumptions

1.1 Problem Statement

We consider a dynamic system which admits a finite state-space representation and whose dynamics are described by an equation which involves uncertain constant parameters. We are concerned with the design of a dynamic state-feedback controller which ensures, in spite of that uncertainty, that solutions of the closed-loop system are bounded and their x-components converge to a desired set point.

Example: **(1)**
Consider the following one-dimensional system:

$$\dot{x} = p^* x^2 + u , \tag{2}$$

where p^* is a constant parameter. Would the value of p^* be known, we could use the following linearizing control law to globally stabilize the origin of the closed-loop system:

$$u = -p^* x^2 - x . \tag{3}$$

When only an approximate value $\widehat{p}$ of p^* is known and is used in the control law (3), we obtain the following closed-loop system:

$$\dot{x} = -x + (p^* - \widehat{p})\, x^2 . \tag{4}$$

This system (4) has two equilibrium points:

1. $x = 0$, which is exponentially stable,
2. $x = \dfrac{1}{p^* - \widehat{p}}$, which is exponentially unstable.

Hence, as long as $\widehat{p}$ is not exactly equal to p^*, the global asymptotic stability of $x = 0$ is lost. We notice also that the simple linear control

$$u = -x \tag{5}$$

gives exactly the same qualitative behavior.

Assume now that we apply the following dynamic controller:

$$\begin{aligned} \dot{\widehat{p}} &= x^3 \\ u &= -\widehat{p}\, x^2 - x\,. \end{aligned} \tag{6}$$

This is the linearizing controller where instead of the true value p^* of the parameter, we use an on-line updated estimate $\widehat{p}$. The corresponding closed-loop system we get is:

$$\begin{aligned} \dot{\widehat{p}} &= x^3 \\ \dot{x} &= -x + (p^* - \widehat{p})\, x^2\,. \end{aligned} \tag{7}$$

To study the stability, we consider the function:

$$W(x, \widehat{p}) = \frac{1}{2}\left(x^2 + (\widehat{p} - p^*)^2\right). \tag{8}$$

Its time derivative along the solutions of (7) is:

$$\dot{W} = -x^2\,. \tag{9}$$

It follows that any solution of the closed-loop system (7) is bounded and:

$$\lim_{t\to\infty} x(t) = 0\,. \tag{10}$$

Therefore, the convergence of x to 0 is restored. □

In this example, the parameter p^* enters linearly in the dynamic equation (2). This assumption is fundamental throughout the paper. It is formalized as follows:

Assumption Λ-LP (Λ-Linear Parameterization) (11)
We can find a set of measured *coordinates x in $\mathbb{R}^n$ such that, given an integer k and a C^1 function $\Lambda : \mathbb{R}^n \times \mathbb{R}_+ \to \mathcal{M}_{kn}(\mathbb{R})$, there exist an integer l, two C^1 functions:*

$$a : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\,, \qquad A : \mathbb{R}^n \times \mathbb{R}^m \to \mathcal{M}_{nl}(\mathbb{R})\,,$$

and an unknown *parameter vector p^* in $\mathbb{R}^l$ such that:*

1. *the functions $\Lambda(x,t)a(x,u)$ and $\Lambda(x,t)A(x,u)$ are* known*, i.e., can be evaluated,*
2. *the dynamics of the system to be controlled are described by:*

$$\dot{x} = a(x,u) + A(x,u)\, p^*\,, \tag{12}$$

where u is the input vector in $\mathbb{R}^m$.

To deal with the case where a and A are affine in u, it is useful to introduce the following notation:

$$\begin{aligned} a(x,u) &= a_0(x) + \sum_{i=1}^{m} u_i\, a_i(x) & A(x,u) &= A_0(x) + \sum_{i=1}^{m} u_i\, A_i(x) \\ &\stackrel{\text{def}}{=} a_0(x) + u \odot b(x)\,, & &\stackrel{\text{def}}{=} A_0(x) + u \odot B(x)\,. \end{aligned} \tag{13}$$

Example: Introduction to System (17) **(14)**

Clearly, in the case of the system (2), if we choose:

$$\Lambda(x,t) \;=\; 1\,, \tag{15}$$

assumption Λ-LP is satisfied with the known functions:

$$a(x,u) \;=\; u\,, \quad A(x,u) \;=\; x^2\,. \tag{16}$$

Let us now consider the following two-dimensional system:

$$\begin{aligned} \dot{x}_1 &= x_1^2\left(x_2^2 + c_1 + c_2\,u\right) \\ \dot{x}_2 &= -c_2\,x_2^2\,(x_2 + c_3\,x_1)\,, \end{aligned} \tag{17}$$

where c_1, c_2 and c_3 are three unknown real numbers. Depending on our choice for the function Λ, we get different parameterizations.

– A first possible choice is, with $k = 2$:

$$\Lambda(x_1,x_2,t) \;=\; \begin{pmatrix} 1\;0 \\ 0\;1 \end{pmatrix}. \tag{18}$$

Then Λ-LP is met with $l = 3$, the parameter vector:

$$p^* \;=\; (c_1\,,\, c_2\,,\, c_2c_3)^{\mathrm{T}}\,, \tag{19}$$

and the known functions:

$$a(x_1,x_2) \;=\; \begin{pmatrix} x_1^2\,x_2^2 \\ 0 \end{pmatrix}, \qquad A(x_1,x_2,u) \;=\; \begin{pmatrix} x_1^2 & x_1^2\,u & 0 \\ 0 & -x_2^3 & x_2^2 x_1 \end{pmatrix}. \tag{20}$$

– Another possibility is, with $k = 1$,

$$\Lambda(x_1,x_2,t) \;=\; (x_1\;\, 0)\,. \tag{21}$$

Assumption Λ-LP holds with $l = 2$, the parameter vector:

$$p^* \;=\; (c_1 \quad c_2)^{\mathrm{T}}\,, \tag{22}$$

and the functions:

$$a(x_1,x_2) \;=\; \begin{pmatrix} x_1^2\,x_2^2 \\ 0 \end{pmatrix}, \qquad A(x_1,x_2,u) \;=\; \begin{pmatrix} x_1^2 & x_1^2\,u \\ 0 & -x_2^2\,(x_2 + c_3\,x_1) \end{pmatrix}. \tag{23}$$

Here the function A is unknown since it involves the unknown constant c_3. But both $\Lambda\,a$ and $\Lambda\,A$ are known. In particular, the choice (21) for the function Λ implies that we pay attention to the dynamics of x_1 only and disregard the dynamics of x_2. □

This example illustrates that in order to find a state-space representation and a parameterization and to choose a function Λ satisfying assumption Λ-LP, it may be useful not to work with the a priori given, say physical, coefficients. First, the knowledge of these coefficients may not be relevant to the Stabilization problem. As emphasized by Sastry and Kokotovic [28], this is in particular the case if, by looking at these coefficients as disturbances, they can be rejected. For the system (17) in Example (14), we will see in Example (49) that the coefficient c_3 is irrelevant to the stabilization. Second, it may also be useful to change the a priori given coordinates and define the parameters as functions of the coefficients. For instance, in Example (14), we may choose $p_3 = c_2 c_3$. This trick is classically used in robotics (see [14]). It has also been used by Sastry and Isidori [29] for input-output linearization of systems with relative degree larger than one and by Kanellakopoulos, Kokotovic and Middleton [10] for adaptive dynamic output feeeback.

Example: Introduction to System (25) (24)
Consider a system whose dynamics is described by the following second-order differential equation:

$$\ddot{y} \;=\; u \,+\, L(y)\, p^* \,, \tag{25}$$

where y is a measured output, u is the input and L is a known smooth function. A straightforward state-space representation with $(y\,,\,\dot{y})$ as the state vector is not appropriate for satisfying assumption Λ-LP, since $\dot{y}$ is not measured. Instead, let us introduce the following filtered quantities:

$$\begin{aligned} &\ddot{y}_{\rm f} + \dot{y}_{\rm f} + y_{\rm f} = y && y_{\rm f}(0) = \dot{y}_{\rm f}(0) = 0 \\ &\ddot{u}_{\rm f} + \dot{u}_{\rm f} + u_{\rm f} = u && u_{\rm f}(0) = \dot{u}_{\rm f}(0) = 0 \\ &\ddot{L}_{\rm f} + \dot{L}_{\rm f} + L_{\rm f} = L && L_{\rm f}(0) = \dot{L}_{\rm f}(0) = 0\,. \end{aligned} \tag{26}$$

Then, (25) can be rewritten with the following non-minimal state-space representation, with $\dot{\ }$ denoting differentiation with respect to time:

$$\dot{x} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & p^* & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} x \;+\; \begin{pmatrix} 0 \\ \delta(t) \\ 0 \\ u \\ 0 \\ L(y) \end{pmatrix} \tag{27}$$

$$\begin{aligned} y &= \dot{y}_{\rm f} + y_{\rm f} + u_{\rm f} + L_{\rm f}\, p^* + \delta(t) \\ \dot{y} &= \dot{y}_{\rm f} + u_{\rm f} + \dot{u}_{\rm f} + \left(L_{\rm f} + \dot{L}_{\rm f}\right) p^* + \dot{\overline{\delta(t)}} + \delta(t)\,, \end{aligned} \tag{28}$$

where x is the following state vector:

$$x \;=\; \left(y_{\rm f}\ \dot{y}_{\rm f}\ u_{\rm f}\ \dot{u}_{\rm f}\ L_{\rm f}\ \dot{L}_{\rm f}\right)^{\rm T}, \tag{29}$$

and $\delta(t)$ is the solution of:

$$\ddot{\delta} + \dot{\delta} + \delta = 0 , \qquad \text{with} \quad \delta(0) = y(0) , \; \dot{\delta}(0) = \dot{y}(0) - y(0) . \tag{30}$$

Since x can be obtained from the knowledge of y and u only, assumption Λ-LP is satisfied up to the presence of the exponentially decaying time function δ by choosing:

$$\Lambda(x,t) = (0\ 1\ 0\ 0\ 0\ 0) . \tag{31}$$

The presence of δ implies that our forthcoming results will not apply in a straightforward manner. In each case we will have to study the effects of this term. □

According to assumption Λ-LP, the only knowledge we have about the system to be controlled is that it is a member of a linearly parameterized family of systems whose dynamics satisfy the following equation, denoted by (S_p) in the sequel:

$$\dot{x} = a(x,u) + A(x,u)\,p . \tag{S_p}$$

In this case, we formalize the Adaptive Stabilization problem as follows:

Adaptive Stabilization Problem: Find an integer ν and two functions

$$\mu_1 : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^\nu , \qquad \mu_2 : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^m ,$$

such that there exists an open subset $\mathcal{D}$ of $\mathbb{R}^n \times \mathbb{R}^\nu$ with the following property:
The solutions $(x(t), \chi(t))$ of the system composed of the system (S_{p^*}) to be controlled and the following dynamic *state-feedback* controller:

$$\begin{aligned} \dot{\chi} &= \mu_1(x, \chi) \\ u &= \mu_2(x, \chi) , \end{aligned} \tag{32}$$

with $(x(0), \chi(0))$ in $\mathcal{D}$,

(AS1) are well-defined, unique and bounded on $[0, +\infty)$,
(AS2) have the property

$$\lim_{t\to\infty} x(t) = \mathcal{E} , \tag{33}$$

where $\mathcal{E}$ is a desired set point for x, which may depend on p^*.

A typical illustration of this problem has been given in Example (1). It turns out that, in all the solutions to this problem which are proposed in this paper, part of the state χ of the controller can be considered as an estimate $\widehat{p}$ of the unknown parameter vector p^*. This motivates the adjective "adaptive".

Several solutions to the Adaptive Stabilization problem have been proposed in the literature under particular assumptions. Extending the work of Pomet [19], we present here a framework allowing us to unify and generalize most of these solutions.

1.2 Connection with the Error Feedback Regulator Problem

When, in equation (S_p), the function A does not depend on u, i.e., $A(x,u) = A_o(x)$, and a is affine in u (see (13)), the Adaptive Stabilization problem described above has similarities with the Error Feedback Regulator problem as stated by Isidori for systems of the form [6, Sect. 7.2]:

$$\begin{aligned} \dot{x} &= a_0(x) \,+\, u \odot b(x) \,+\, A_o(x)\, p \\ \dot{p} &= s(p) \\ y &= h(x) \,+\, q(p)\,, \end{aligned} \tag{34}$$

where p in $\mathbb{R}^l$ is an unmeasured disturbance while y in $\mathbb{R}^k$ is a measured output signal.

Error Feedback Regulator Problem: Find an integer ν and two functions

$$\mu_1 \,:\, \mathbb{R}^k \times \mathbb{R}^\nu \,\to\, \mathbb{R}^\nu\,, \qquad \mu_2 \,:\, \mathbb{R}^\nu \,\to\, \mathbb{R}^m\,,$$

such that:

(EFR1) the equilibrium point of

$$\begin{aligned} \dot{x} &= a_0(x) \,+\, \mu_2(\chi) \odot b(x) \,+\, A_0(x)\, p^* \\ \dot{\chi} &= \mu_1(h(x), \chi) \end{aligned} \tag{35}$$

is asymptotically stable in the first approximation,

(EFR2) there exists an open subset $\mathcal{D}$ of $\mathbb{R}^n \times \mathbb{R}^\nu$ such that the solutions of the system (34) controlled by the following dynamic *output* feedback:

$$\begin{aligned} \dot{\chi} &= \mu_1(y, \chi) \\ u &= \mu_2(\chi)\,, \end{aligned} \tag{36}$$

with initial condition in this set $\mathcal{D}$, have the property

$$\lim_{t \to \infty} y(t) \;=\; \mathcal{E}\,, \tag{37}$$

where $\mathcal{E}$ is a desired set point for y, independent of p.

It would appear that the family of systems (S_p) dealt with in our Adaptive Stabilization problem is a subclass of the systems (34) of the Error Feedback Regulator problem. This family is obtained by imposing a constant disturbance, i.e.,

$$\dot{p} \;=\; s(p) \;=\; 0\,, \tag{38}$$

and a measured state, i.e.,

$$h(x) \;=\; x \qquad \text{and} \qquad q(p) \;=\; 0\,. \tag{39}$$

Isidori gives a solution to the Error Feedback Regulator problem in [6, Theorem 7.2.10]. It applies to systems (S_p) under the following assumptions, using the notation (13):

(A1) There exists a C^2 function $u_0(p)$ defined on an open neighborhood of p^* such that:

$$a_0(\mathcal{E}) \,+\, u_0(p) \odot b(\mathcal{E}) \,+\, A_0(\mathcal{E})\, p \;=\; 0\,, \tag{40}$$

(A2) The $n \times l$ matrix $A_0(\mathcal{E})$ has rank l,

(A3) The pair $\left(\frac{\partial a_0}{\partial x}(\mathcal{E}) + u_0(p^*) \odot \frac{\partial b}{\partial x}(\mathcal{E}) + \frac{\partial A_0}{\partial x}(\mathcal{E})\, p^*\,,\ b(\mathcal{E})\right)$ is stabilizable.

Assumption (A1) is quite natural if $\mathcal{E}$ is not allowed to depend on p. It states only the existence of a control, smoothly depending on p, making $\mathcal{E}$ a set point of (S_p) for all p close enough to p^*. Assumption (A2) is much more restrictive since it imposes that the number l of parameters do not exceed the dimension n of the state and that the matrix $A_0(x)$ cannot degenerate at $x = \mathcal{E}$. Finally, assumption (A3) excludes systems whose linearization has uncontrollable modes associated with pure imaginary eigenvalues and, in particular, systems which are not feedback linearizable.

Example: System (17) Continued (41)

Assumptions (A2) and (A3) are not satisfied by the following system (i.e., system (17) with $c_2 = 1$ to make A independent of u):

$$\begin{aligned} \dot{x}_1 &= x_1^2\left(x_2^2\, p_1 + u\right) \\ \dot{x}_2 &= -\,x_2^2\,(x_2 + p_2\, x_1)\ . \end{aligned} \tag{42}$$

Indeed, $A(\mathcal{E})$, $\frac{\partial a}{\partial x}(\mathcal{E}, u)$ and $\frac{\partial A}{\partial x}(\mathcal{E})$ are all zero. □

In fact, assumptions (A2) and (A3) follow from the strong requirement EFR1 of asymptotic stability. In this paper, it is precisely in order to be able to deal with systems which may not satisfy assumptions (A2) or (A3), such as system (42), that, instead of requirement EFR1, we ask, in the Adaptive Stabilization problem, for the less stringent Lagrange stability requirement AS1.

1.3 Assumptions

Our counterparts of assumptions (A1), (A2) and (A3) are the following assumptions on the particular system (S_{p^*}):

Let Π be an open subset of $\mathbb{R}^l$ and Ω be an open neighborhood of $\mathcal{E}$ in $\mathbb{R}^n$. There exist two known *functions:*

$$V\ :\ \Omega \times \Pi\ \to\ \mathbb{R}_+ \textit{ of class } C^2\,, \qquad u_{\mathrm{n}}\ :\ \Omega \times \Pi\ \to\ \mathbb{R}^m \textit{ of class } C^1\,,$$

such that:

Assumption BO (Boundedness Observability) (43)

There exist an open neighborhood Ω_0 of $\mathcal{E}$ in Ω and a strictly positive constant α_0 such that for all real numbers α, $0 < \alpha < \alpha_0$, all compact subsets $\mathcal{K}$ of Π and all vectors $x_0 \in \Omega_0$, we can find a compact subset Γ of Ω such that, for any C^1 time functions $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$ and any solution $x(t)$ of:

$$\dot{x}\ =\ a(x, u(t))\ +\ A(x, u(t))\, p^*\,, \qquad x(0)\ =\ x_0\ \in\ \Omega_0 \tag{44}$$

defined on $[0, T)$, we have the following implication:

$$V(x(t), \widehat{p}(t)) \leq \alpha \textit{ and } \widehat{p}(t)\ \in\ \mathcal{K}\quad \forall\, t\ \in\ [0, T) \quad \Longrightarrow \quad x(t)\ \in\ \Gamma \quad \forall\, t\ \in\ [0, T)\,. \tag{45}$$

Assumption PRS (Pointwise Reduced-Order Stabilizability) (46)
For all (x,p) in $\Omega \times \Pi$, we have:

$$\frac{\partial V}{\partial x}(x,p)\,[a(x,u_{\mathrm{n}}(x,p)) + A(x,u_{\mathrm{n}}(x,p))\,p] \;\leq\; 0\,, \tag{47}$$

where the inequality is strict iff $V(x,p) \neq 0$.

In the sequel, the case where

$$\Omega \;=\; \Omega_0 \;=\; \mathbb{R}^n \qquad \text{and} \qquad \alpha_0 \;=\; +\infty \tag{48}$$

is called the global case.

Example: System (17) Continued (49)
Consider the two-dimensional system (17) in Example (14) which, according to the second parameterization we mentioned, is rewritten as:

$$\begin{aligned} \dot{x}_1 &= x_1^2\left(x_2^2 + p_1 + p_2\,u\right) \\ \dot{x}_2 &= -p_2\,x_2^2\,(x_2 + c_3\,x_1)\,, \end{aligned} \tag{50}$$

where p_2 is known to be strictly positive. We choose:

$$V(x_1,x_2,p_1,p_2) \;=\; \frac{1}{2}\,x_1^2\,, \qquad u_{\mathrm{n}}(x_1,x_2,p_1,p_2) \;=\; -\frac{x_2^2 + p_1 + x_1}{p_2} \tag{51}$$

$$\Omega_0 \;=\; \Omega \;=\; \mathbb{R}^2\,, \qquad \Pi \;=\; \mathbb{R} \times (\mathbb{R}_+ - \{0\})\,, \qquad \alpha_0 = +\infty\,. \tag{52}$$

Notice that the constant c_3 is not involved. This justifies a posteriori the choice of the parameterization.

Assumption PRS is satisfied. Indeed, we get:

$$\frac{\partial V}{\partial x}(x,p)\,[a(x,u_{\mathrm{n}}(x,p)) + A(x,u_{\mathrm{n}}(x,p))\,p] \;=\; -x_1^4\,. \tag{53}$$

Assumption BO holds also if, for any C^1 time function $x_1(t)$, we have:

$$|x_1(t)| \;\leq\; \alpha \quad \forall\, t \,\in\, [0,T) \qquad \Longrightarrow \qquad |x_2(t)| \;\leq\; \gamma \quad \forall\, t \,\in\, [0,T)\,, \tag{54}$$

where $x_2(t)$ is a solution of:

$$\dot{x}_2 \;=\; -p_2^*\,x_2^2\,(x_2 + c_3\,x_1(t))\,, \tag{55}$$

and γ is a positive function of α and $x_2(0)$. To prove this implication, we notice that ${p_2}^* \,>\, 0$ implies:

$$|x_2(t)| \;>\; |c_3|\,|x_1(t)| \qquad \Longrightarrow \qquad \dot{\overline{x_2^2(t)}} \;<\; 0\,. \tag{56}$$

Hence, with (54), we can choose:

$$\gamma \;=\; \max\left\{|x_2(0)|\,,\,|c_3|\,\alpha\right\}\,. \tag{57}$$

□

This example illustrates two points of our assumptions:

1. The parameter vector p must in some cases be constrained to lie in Π, an open set strictly contained in $\mathbb{R}^l$. Indeed, here, for $p_2 = 0$, u_{n} is not defined.

2. The function V looks like a Lyapunov function in x, but actually it may not be radially unbounded in x. For instance, in Example (49), x_2 may go to infinity without V going to infinity. The radial unboundedness of a Lyapunov function $V(x(t),t)$ guarantees that the magnitude of $V(x(t),t)$ at time t gives a bound on the norm of the *full* state vector $x(t)$ for the same time t. In contrast, as illustrated by Example (49), the magnitude of $V(x(t),p)$, given by assumption BO, at time t gives a bound on the norm of only a *part* of the state vector $x(t)$ at the same time t, namely $x_1(t)$. What assumption BO actually guarantees is the "observability" of the boundedness of the full state vector x from the "output" function $V(x,p)$, i.e., that if the trajectory $\{V(x(t),p)\}_{t\in[0,T)}$ of the "output" is bounded, then so is the trajectory $\{x(t)\}_{t\in[0,T)}$ of the full state. Then, assumption PRS guarantees the existence of a control law which forces the part of the state vector mentioned above to converge to the corresponding part of the equilibrium point $\mathcal{E}$. Later on, we will add more constraints on the function V, e.g. assumption MC. In order to allow for more possibilities to find such a function V meeting all these requirements, it is useful at this stage to have assumption BO instead of the more standard but more restrictive radial unboundedness.

To guarantee convergence to $\mathcal{E}$ of the whole state vector x and not only of its reduced-order part, we will need also:

Assumption CO (Convergence Observability) (58)

For any bounded C^1 time functions $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$ with $\dot{\widehat{p}}$ also bounded and for any solution $x(t)$ of (44) defined on $[0,+\infty)$, we have the following implication:

$$\left.\begin{array}{c} \lim_{t\to\infty} V(x(t),\widehat{p}(t)) \text{ exists and is zero,} \\ x(t) \text{ is bounded on } [0,+\infty) \\ \text{and} \quad x(t) \in \Omega \quad \forall\, t \in [0,+\infty) \end{array}\right\} \implies \lim_{t\to\infty} x(t) \text{ exists and is equal to } \mathcal{E}.$$

Example: System (17) Continued (59)

Assumption CO is satisfied for the system (17) rewritten as (50) with the choice (51). This is a consequence of the uniform asymptotic stability of the zero solution of (see Lakshmikantham and Leela [11, Theorem 3.8.3]):

$$\dot{x}_2 = -p_2\, x_2^3\,. \tag{60}$$

Indeed, since x_1 is a component of a solution of an ordinary differential equation, the convergence of $x_1^2(t)$ to 0, implies the boundedness:

$$|x_1(t)| \le \alpha \quad \forall\, t\,. \tag{61}$$

Then, since assumption BO is satisfied,

$$|x_2(t)| \le \gamma = \max\{|x_2(0)|\,,\, |c_3|\,\alpha\} \quad \forall\, t\,. \tag{62}$$

Now, the convergence of $x_1^2(t)$ to 0 implies also that for all $\varepsilon > 0$ there exists a time T such that:

$$|x_1(t)| \le \varepsilon \qquad \forall\, t \ge T\,. \tag{63}$$

And, with (55), we have:

$$\overline{x_2^2(t)}^{\cdot} \leq -p_2 \left(x_2^4(t) - |c_3|\, \varepsilon \, |x_2(t)|^3 \right) \quad \forall t \geq T\,. \tag{64}$$

It follows that:

$$|x_2(t)| \geq \varepsilon\,(1 + |c_3|) \qquad \rightarrow \qquad \overline{x_2^2(t)}^{\cdot} \leq -\varepsilon^4\,. \tag{65}$$

Therefore, with (62), defining $\tau(\varepsilon)$ by:

$$\tau = \frac{1}{\varepsilon^4} \left(\gamma^2 - \varepsilon^2\,(1 + |c_3|)^2 \right), \tag{66}$$

we have established:

$$\forall \varepsilon > 0\ \exists T, \exists \tau \,:\, \forall t \geq T + \tau\,,\ |x_2(t)| \leq \varepsilon\,(1 + |c_3|)\,. \tag{67}$$

This is CO. □

This Example illustrates a typical fact about assumption CO: it is usually difficult to prove that it holds. Above we used a Total Stability argument, here is another example invoking Barbălat's Lemma [26, p.211]:

Example: **(68)**
Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 + p^* \\ \dot{x}_2 &= u\,. \end{aligned} \tag{69}$$

We wish to stabilize the equilibrium point $x_1 = 0, x_2 = -p^*$. Assumptions BO and PRS are satisfied when we choose:

$$V(x_1, x_2, p) = x_1^2 + (x_2 + p + x_1)^2\,, \quad u_{\mathrm{n}}(x_1, x_2, p) = -2\,(x_2 + p + x_1)\,. \tag{70}$$

To check that CO is also met, we follow the same steps as the ones proposed by Kanellakopoulos, Kokotovic and Marino for the proof of [9, Theorem 1]. Clearly, if $V(x_1(t), x_2(t), \widehat{p}(t))$ tends to 0, the same holds for $x_1(t)$ and $x_2(t) + x_1(t) + \widehat{p}(t)$. To obtain our conclusion, it is sufficient to prove that $\overline{x_1(t)}^{\cdot}$ tends to 0. Indeed, in such a case, the first equation in (69) implies in this case that $x_2(t) + p^*$ tends to 0. Since we have:

$$\lim_{t \to +\infty} x_1(t) = 0 = x_1(0) + \lim_{t \to +\infty} \int_0^t \overline{x_1(s)}^{\cdot}\, ds\,, \tag{71}$$

from Barbălat's Lemma, $\overline{x_1(t)}^{\cdot}$ tends to 0 if this time function is uniformly continuous. This is indeed the case since its time derivative is:

$$\overline{x_1(t)}^{\cdot\cdot} = \overline{x_2(t)}^{\cdot} = u(t)\,, \tag{72}$$

where, by assumption, the time function $u(t)$ is bounded. □

Assumptions BO, PRS and CO are weaker than (A1), (A2) and (A3). Indeed, it follows from linear systems theory [7] and Total Stability theorems [11] that assumptions (A1) and (A3) imply the existence of:

- Ω, an open neighborhood of $\mathcal{E}$,
- Π, an open neighborhood of p^*,

- P, an $n \times n$ positive definite matrix,
- C, an $n \times m$ matrix, and
- k, a strictly positive constant,

such that, by letting:

$$V(x,p) = V(x) = (x-\mathcal{E})^{\mathrm{T}} P\,(x-\mathcal{E})\ , \quad u_{\mathrm{n}}(x,p) = -C\,(x-\mathcal{E}) + u_0(p)\,, \tag{73}$$

we have:

$$\frac{\partial V}{\partial x}(x)\,[a_0(x) + u_{\mathrm{n}}(x,p)\odot b(x) + A_0(x)\,p] \,\leq\, k\,V(x)\,, \quad \forall\,(x,p)\,\in\,\Omega\times\Pi\,. \tag{74}$$

This implies that assumption PRS is satisfied. Also, the function V is positive definite and radially unbounded. Hence, assumptions BO and CO hold with $\Omega_0 = \Omega$ and α_0 the largest positive real number α such that:

$$V(x) \,<\, \alpha \qquad \Longrightarrow \qquad x \,\in\, \Omega\,. \tag{75}$$

Finally, note that (A2) is not needed.

If the value of the parameter vector p^* were known, assumptions BO, PRS and CO would be sufficient to guarantee the stabilizability of $\mathcal{E}$. This is made precise as follows:

Proposition **(76)**
Let assumptions BO and PRS hold, p^* be in Π and the control $u_{\mathrm{n}}(x,p^*)$ be applied to the system (S_{p^*}). Under these conditions, all the solutions $x(t)$ with initial condition $x(0)$ in Ω_0 and satisfying $V(x(0),p^*) \,<\, \alpha_0$ are well-defined on $[0,+\infty)$, unique, bounded and:

$$\lim_{t\to\infty} V(x(t),p^*) \,=\, 0\,. \tag{77}$$

If, moreover, assumption CO holds, then $x(t)$ converges to $\mathcal{E}$.

Proof. The system we consider is:

$$\dot{x} \,=\, a(x,u_n(x,p^*)) + A(x,u_n(x,p^*))p^*\,. \tag{78}$$

With p^* fixed in Π, this is an autonomous system with its right-hand side continuously differentiable in the open subset Ω of $\mathbb{R}^n$. Hence, for any initial condition $x(0)$ in $\Omega_0 \subset \Omega$, there exists a unique solution $x(t)$ of (78) in Ω. It is a continuously differentiable time function (and so is u_{n}) defined on a right maximal interval $[0,T)$, with T a strictly positive (possibly infinite) real number. Let us prove by contradiction that $T=+\infty$ if $V(x(0),p^*) \,<\, \alpha_0$. Assume the contrary. From the theorem on continuation of solutions [5, Theorem I.2.1], $x(t)$ tends to the boundary of Ω as t tends to T. But since:

$$x(t) \,\in\, \Omega\,,\ \forall\,t\,\in\,[0,T) \qquad \text{and} \qquad p^* \,\in\, \Pi\,, \tag{79}$$

we may use (47) in assumption PRS and conclude:

$$\dot{\overline{V(x(t),p^*)}} \,\leq\, 0 \quad \forall\,t\,\in\,[0,T)\,. \tag{80}$$

This yields:

$$V(x(t),p^*) \,\leq\, V(x(0),p^*) \,<\, \alpha_0 \quad \forall\,t\,\in\,[0,T)\,. \tag{81}$$

Then, from assumption BO, we know there exists a compact subset Γ of Ω, depending on p^*, $x(0)$ and $V(x(0),p^*)$, such that:

$$x(t) \,\in\, \Gamma\,,\ \forall\,t\,\in\,[0,T)\,. \tag{82}$$

Since the set Γ is compact, it is strictly contained in the open set Ω. This establishes the contradiction and the fact that the time functions $x(t)$ and $u(t) = u_{\mathrm{n}}(x(t), p^*)$ are bounded on $[0, +\infty)$.

Now, from (47), we have:

$$\begin{aligned} \overline{V(x(t), p^*)}^{\cdot} &< 0 \quad \forall t : V(x(t), p^*) \neq 0 \\ &= 0 \quad \forall t : V(x(t), p^*) = 0 \,. \end{aligned} \tag{83}$$

Since the function V is nonnegative, this implies (77).

Finally, convergence to $\mathcal{E}$ of $x(t)$ is a straightforward consequence of assumption CO. □

In the following we will show that even when the value of p^* is unknown, in which case we cannot implement $u_{\mathrm{n}}(x, p^*)$, the Adaptive Stabilization problem can be solved if extra assumptions are added. In Section 2, a solution will be obtained from the Lyapunov design with the assumption that a so-called matching condition is satisfied. In Section 3, other solutions will be obtained from an estimation approach. They will require a stronger version of assumption PRS and that either a matching condition or some growth conditions on the nonlinearities be satisfied.

2 Lyapunov Design

In a famous paper, Parks [18] suggested a very efficient way of getting a controller for the linearly parameterized family of systems (S_p). The idea is to use the control:

$$u = u_{\mathrm{n}}(x, \widehat{p}) \tag{84}$$

with the time function $\widehat{p}$ selected so that a positive definite radially unbounded function of x and $\widehat{p}$ be decaying. Even though here the function V is not radially unbounded, let us pursue this idea and compute the time derivative of

$$W(x, \widehat{p}) = V(x, \widehat{p}) + \frac{1}{2} \|\widehat{p} - p^*\|^2 \tag{85}$$

along the solutions of (S_{p^*})-(84). Using (47) in assumption PRS, we get:

$$\begin{aligned} \dot{W} &= \frac{\partial V}{\partial x}(x, \widehat{p}) \left[a(x, u_{\mathrm{n}}(x, \widehat{p})) + A(x, u_{\mathrm{n}}(x, \widehat{p}))\, p^* \right] \\ &\qquad + \left(\frac{\partial V}{\partial p}(x, \widehat{p}) + [\widehat{p} - p^*]^{\mathrm{T}} \right) \dot{\widehat{p}} \\ &= \frac{\partial V}{\partial x}(x, \widehat{p}) \left[a(x, u_{\mathrm{n}}(x, \widehat{p})) + A(x, u_{\mathrm{n}}(x, \widehat{p}))\, \widehat{p} \right] \\ &\qquad + \left[-\frac{\partial V}{\partial x}(x, \widehat{p})\, A(x, u_{\mathrm{n}}(x, \widehat{p})) + \dot{\widehat{p}}^{\mathrm{T}} \right] [\widehat{p} - p^*] + \frac{\partial V}{\partial p}(x, \widehat{p}) \dot{\widehat{p}} \\ &\leq \left[-\frac{\partial V}{\partial x}(x, \widehat{p})\, A(x, u_{\mathrm{n}}(x, \widehat{p})) + \dot{\widehat{p}}^{\mathrm{T}} \right] [\widehat{p} - p^*] + \frac{\partial V}{\partial p}(x, \widehat{p}) \dot{\widehat{p}} \,. \end{aligned} \tag{86}$$

2.1 Case: V Independent of p, i.e., $V(x,p) = V(x)$

It follows that in the particular case where the function V does not depend on p, by choosing:

$$\dot{\widehat{p}} = \left[\frac{\partial V}{\partial x}(x)\, A(x, u_{\mathrm{n}}(x, \widehat{p}))\right]^{\mathrm{T}}, \qquad \widehat{p}(0) \in \Pi\,, \tag{87}$$

we are guaranteed that W remains bounded and in particular:

$$V(x(t)) \leq V(x(0)) + \frac{1}{2}\, \|\widehat{p}(0) - p^*\|^2 \,. \tag{88}$$

However, this boundedness property is not sufficient, since to use assumptions BO and PRS, we must check that for all t:

1. the following inequality is satisfied:

$$V(x(t)) < \alpha_0\,, \tag{89}$$

2. the following membership property is satisfied:

$$\widehat{p}(t) \in \Pi\,. \tag{90}$$

It is straightforward to see that, if $x(t)$ and $\widehat{p}(t)$ are continuous functions of t and the following function W:

$$W(x(t), \widehat{p}(t)) = \frac{\alpha_1\, V(x(t))}{\alpha_1 - V(x(t))} + \frac{1}{2}\, \|\widehat{p}(t) - p^*\|^2 \tag{91}$$

is positive and bounded for all t, then, necessarily, $V(x(t))$ is strictly smaller than α_1 for all t. Consequently, (89) is satisfied for all t if we can ensure that the modified function $W(x(t), \widehat{p}(t))$ (91) is positive and bounded for all t and the constant α_1 is chosen smaller than or equal to α_0.

In order to meet the membership property (90), we shall constrain $\widehat{p}$ to remain in a closed convex subset of Π by projection of $\dot{\widehat{p}}$. For this, we need the following property of the set Π:

Assumption ICS (Imbedded Convex Sets) **(92)**
There exists a known *convex* C^2 *function* $\mathcal{P}$ *from* $\mathbb{R}^l$ *to* $\mathbb{R}$ *such that:*

1. *for each real number* λ *in* $[0\,,\,1]$*, the set:*

$$\Pi_\lambda = \{p \mid \mathcal{P}(p) \leq \lambda\} \tag{93}$$

is contained in Π*,*

2. *there exists a strictly positive constant* d *such that:*

$$\left\|\frac{\partial \mathcal{P}}{\partial p}(p)\right\| \geq d \qquad \forall\, p \in \{p \mid 0 \leq \mathcal{P}(p)\}\,, \tag{94}$$

3. *the parameter vector* p^* *of the system to be actually controlled satisfies:*

$$\mathcal{P}(p^*) < 0 \qquad \textit{and} \qquad D^* \stackrel{\mathrm{def}}{=} \mathrm{dist}\,(p^*\,, \{p \mid \mathcal{P}(p) = 0\}) > 0\,. \tag{95}$$

Example: System (17) Continued **(96)**

For the system (17) rewritten as (50) in Example (49), the set Π is defined by $p_2 > 0$. Then, by choosing:

$$\mathcal{P}(p_1, p_2) \;=\; 2\left(1 - \frac{p_2}{\varepsilon}\right), \tag{97}$$

with $\varepsilon > 0$, this set Π satisfies assumption ICS if $p_2^* > \varepsilon$.

More generally, consider the case where the set Π is:

$$p = (p_1, \ldots, p_l)^{\mathrm{T}} \in \Pi \quad \Longleftrightarrow \quad |p_i - \rho_i| < \sigma_i \;, \quad \forall\, i \in \{1, \ldots, l\}, \tag{98}$$

with ρ_i and σ_i some given real numbers. To meet assumption ICS we may choose the function $\mathcal{P}$ as:

$$\mathcal{P}(p) = \frac{2}{\varepsilon}\left[\sum_{i=1}^{l}\left|\frac{p_i - \rho_i}{\sigma_i}\right|^q - 1 + \varepsilon\right], \tag{99}$$

with $0 < \varepsilon < 1$ and $q \geq 2$ two real numbers. In this case, we get:

$$\Pi_\lambda \;=\; \left\{ p \;\middle|\; \sum_{i=1}^{l}\left|\frac{p_i - \rho_i}{\sigma_i}\right|^q \;\leq\; 1 - \varepsilon\left(1 - \frac{\lambda}{2}\right)\right\}, \tag{100}$$

and the set Π_λ, for $\lambda = 1$, approaches Π when ε decreases and q increases. □

Assumption ICS allows us to define the closed convex subset Π_1 of Π as:

$$\Pi_1 \;=\; \{p \mid \mathcal{P}(p) \;\leq\; 1\}\;, \tag{101}$$

and the function Proj as:

$$\mathrm{Proj}(M, p, y) = \begin{cases} y & \text{if } \mathcal{P}(p) \leq 0 \;\text{ or }\; \frac{\partial \mathcal{P}}{\partial p}(p)\, y \leq 0 \\ y - \frac{\mathcal{P}(p)\, \frac{\partial \mathcal{P}}{\partial p}(p) y}{\frac{\partial \mathcal{P}}{\partial p}(p) M \frac{\partial \mathcal{P}}{\partial p}(p)^{\mathrm{T}}}\, M\, \frac{\partial \mathcal{P}}{\partial p}(p)^{\mathrm{T}} & \text{if } \mathcal{P}(p) > 0 \text{ and } \frac{\partial \mathcal{P}}{\partial p}(p)\, y > 0\,, \end{cases} \tag{102}$$

where M is a symmetric positive definite $l \times l$ matrix. Namely, $\mathrm{Proj}(M, p, y)$ is equal to y if p belongs to the set $\{\mathcal{P}(p) \leq 0\}$. In the set $\{0 \leq \mathcal{P}(p) \leq 1\}$, it substracts a vector M-normal to the boundary $\{\mathcal{P}(p) = \lambda\}$, so that we get a smooth transformation from the original vector field for $\lambda = 0$ to an inward or tangent vector field for $\lambda = 1$. We have the following technical properties proved in Appendix A:

Lemma **(103)**

Let $\mathcal{M}$ be the open set of symmetric positive definite $l \times l$ matrices. If assumption ICS holds, then:

1. *The function* $\mathrm{Proj}(M, p, y) \,:\, \mathcal{M} \times \Pi \times \mathbb{R}^l \to \mathbb{R}^l$ *is locally Lipschitz-continuous.*
2. $\mathrm{Proj}(M, p, y)^{\mathrm{T}} M^{-1} \mathrm{Proj}(M, p, y) \;\leq\; y^{\mathrm{T}} M^{-1} y \qquad \forall\, p \in \Pi_1$
3. $\frac{\partial \mathcal{P}}{\partial p}(p)(p - p^*) \;\geq\; D^* \left\|\frac{\partial \mathcal{P}}{\partial p}(p)\right\| \qquad \forall\, p \,:\, \mathcal{P}(p) \geq 0$ *with D^* defined in (95),*
4. $(p - p^*)^{\mathrm{T}} M^{-1} \mathrm{Proj}(M, p, y) \;\leq\; (p - p^*)^{\mathrm{T}} M^{-1} y$
5. *Let $(M, y) \,:\, \mathbb{R}_+ \to \mathcal{M} \times \mathbb{R}^l$ be a C^1 time function. On their domain of definition, the solutions of:*

$$\dot{\widehat{p}} \;=\; \mathrm{Proj}\left(M(t)\,,\, \widehat{p}\,,\, y(t)\right) \qquad \widehat{p}(0) \;\in\; \Pi_1 \tag{104}$$

satisfy $\widehat{p}(t) \;\in\; \Pi_1$.

We are now ready to propose the following dynamic controller to solve the Adaptive Stabilization problem when the function V does not depend on p:

$$\begin{aligned} \dot{\widehat{p}} &= \mathrm{Proj}\left(I\,,\,\widehat{p}\,,\ \frac{{\alpha_1}^2}{(V(x)-\alpha_1)^2}\left[\frac{\partial V}{\partial x}(x)\,A(x,u_{\mathrm{n}}(x,\widehat{p}))\right]^{\mathrm{T}}\right) \\ u &= u_{\mathrm{n}}(x\,,\,\widehat{p})\,, \end{aligned} \tag{105}$$

where $\widehat{p}(0)$ is selected in Π_1 and the matrix M used in the function Proj is I, the identity matrix. We have:

Proposition **(106)**
Let assumptions BO, PRS and ICS hold with a function V not depending on p. Assume also that assumption Λ-LP is satisfied with:

$$\Lambda(x,t) \;=\; \frac{\partial V}{\partial x}(x)\,. \tag{107}$$

If, in (105), α_1 is chosen smaller than or equal to α_0, then all the solutions $(x(t),\widehat{p}(t))$ of (S_{p^*})-(105) with $x(0)\in\Omega_0$ and $V(x(0))<\alpha_1$ are well-defined on $[0,+\infty)$, unique, bounded and:

$$\lim_{t\to\infty} V(x(t)) \;=\; 0\,. \tag{108}$$

It follows that the Adaptive Stabilization problem is solved if assumption CO also holds.

Proof. The system we consider is:

$$\begin{aligned} \dot{x} &= a(x,u_{\mathrm{n}}(x\,,\,\widehat{p})) + A(x,u_{\mathrm{n}}(x\,,\,\widehat{p}))p^* \\ \dot{\widehat{p}} &= \mathrm{Proj}\left(I\,,\,\widehat{p}\,,\ \frac{{\alpha_1}^2}{(V(x)-\alpha_1)^2}\left[\frac{\partial V}{\partial x}(x)\,A(x,u_{\mathrm{n}}(x,\widehat{p}))\right]^{\mathrm{T}}\right). \end{aligned} \tag{109}$$

¿From our smoothness assumptions on the functions a, A, u_{n} and V and with Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{p}) \,\in\, \Omega\times\Pi \qquad \text{and} \qquad V(x) \,<\, \alpha_1\,. \tag{110}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{p}(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite, and satisfying (110) for all t in $[0,T)$. Applying point 5 of Lemma (103), we also know that $\widehat{p}(t)\in\Pi_1$ for all t in $[0,T)$. Then, let us compute the time derivative of W defined in (91) along such a solution. With assumption PRS, we get as in (86):

$$\dot{W} \le \left[-\frac{\alpha_1^2}{(V(x(t))-\alpha_1)^2}\left(\frac{\partial V}{\partial x}(x(t))\,A(x,u_n(x(t),\widehat{p}(t))\right)^{\mathrm{T}} + \dot{\widehat{p}}^{\mathrm{T}}(t)\right](\widehat{p}(t)-p^*)\,, \tag{111}$$

with a strict inequality if $V(x(t))\neq 0$. But, with the expression of $\dot{\widehat{p}}$ and Point 4 of Lemma (103), we get readily:

$$\begin{aligned} \dot{W} &\le 0 \ \text{ if } V(x(t)) = 0 \\ &< 0 \ \text{ if } V(x(t)) \neq 0\,. \end{aligned} \tag{112}$$

It follows that for all t in $[0, T)$:

$$\begin{aligned} V(x(t)) &< \alpha_1 \\ \frac{\alpha_1\, V(x(t))}{\alpha_1 \,-\, V(x(t))} &\leq \frac{\alpha_1\, V(0)}{\alpha_1 \,-\, V(0)} \;+\; \frac{1}{2}\, \|\widehat{p}(0) - p^*\|^2 \;\stackrel{\text{def}}{=}\; \beta \\ \|\widehat{p}(t) - p^*\|^2 &\leq 2\,\beta\,. \end{aligned} \tag{113}$$

Hence, we get:

$$V(x(t)) \;\leq\; \frac{\alpha_1\, \beta}{\alpha_1 \,+\, \beta} \;\stackrel{\text{def}}{=}\; \alpha \;<\; \alpha_1 \;\leq\; \alpha_0\,, \tag{114}$$

and we know that $\widehat{p}(t) \in \mathcal{K}$, where $\mathcal{K}$ is the following compact subset of Π:

$$\mathcal{K} \;=\; \left\{ p \;\middle|\; \|p - p^*\|^2 \;\leq\; 2\,\beta \right\} \bigcap \Pi_1\,. \tag{115}$$

Then, from assumption BO, we know the existence of a compact subset Γ of Ω such that:

$$x(t) \;\in\; \Gamma \quad \forall\, t \;\in\; [0, T)\,. \tag{116}$$

Hence, the solution remains in a compact subset of the open set defined in (110). It follows by contradiction that $T \;=\; +\infty$ and, in particular, that the time functions $x(t)$, $\widehat{p}(t)$, $u(t) \;=\; u_{\text{n}}(x(t), \widehat{p}(t))$ and $\dot{\widehat{p}}(t)$ are bounded on $[0, +\infty)$. Then (108) is a straightforward consequence of (112) and LaSalle's Theorem [5, Theorem X.1.3]. The conclusion follows readily from assumption CO. □

Example: **(117)**

Consider the following system:

$$\dot{x} \;=\; p^*\, x^3 \;+\; (1 - x^2)\, u \tag{118}$$

with p^* positive. We wish to stabilize the equilibrium point $x = 0$. Clearly, even if p^* were known, this would not be possible globally, but only for $x \in (-1\,,\, 1)$. Then let $\Omega = (-1\,,\, 1)$ and choose:

$$V(x) = x^2 \quad \text{and} \quad u_{\text{n}}(x, p) = -\frac{p\, x^3 + x}{1 - x^2}\;. \tag{119}$$

Assumptions Λ-LP, BO, PRS and CO are satisfied with $\alpha_0 = 1$. According to Proposition (106), the following dynamic controller guarantees the convergence of $x(t)$ to 0 for all initial conditions $\widehat{p}(0)$ and $x(0)$, with $|x(0)| < 1$:

$$\begin{aligned} \dot{\widehat{p}} &= \frac{2x^4}{(1 - x^2)^2} \\ u &= -\frac{\widehat{p}\, x^3 + x}{1 - x^2}\;. \end{aligned} \tag{120}$$

□

Compared with Proposition (76), this Proposition (106) states that, for solving the Adaptive Stabilization problem when p^* is unknown, the existence of a function V independent of p is a sufficient condition. This shows that we should look for a control law u_{n} for which we can find such a function V to meet assumption BO and PRS (see [19, Chapter 2]). This is where the fact that V need not be radially unbounded proves to be useful, as we now illustrate:

Example: System (17) Continued **(121)**
For the system (17) rewritten as (50) in Example (49), we have established that assumptions Λ-LP, BO, PRS and ICS hold when Λ, V and u_{n} are chosen as:

$$\Lambda(x_1, x_2, t) = (x_1 \ 0)\,, \qquad V(x_1, x_2, p_1, p_2) = \frac{1}{2}\, x_1^2\,, \tag{122}$$

and:

$$u_{\mathrm{n}}(x_1, x_2, p_1, p_2) = -\frac{x_2^2 + p_1 + x_1}{p_2}\ . \tag{123}$$

By specializing the dynamic controller (105) to this system (50) and choosing $\alpha_1 = +\infty$ and $\mathcal{P}$ as in (97), the following controller solves the Adaptive Stabilization problem:

$$\begin{aligned}
\dot{\widehat{p}}_1 &= x_1^3 \\
\dot{\widehat{p}}_2 &= -\frac{x_1^3(x_2^2 + \widehat{p}_1 + x_1)}{\widehat{p}_2} && \text{if } \widehat{p}_2 \geq \varepsilon \ \text{ or } \ \frac{x_1^3(x_2^2 + \widehat{p}_1 + x_1)}{\widehat{p}_2} \leq 0 \\
&= \left(1 - 2\frac{\widehat{p}_2}{\varepsilon}\right) \frac{x_1^3(x_2^2 + \widehat{p}_1 + x_1)}{\widehat{p}_2} && \text{if } \widehat{p}_2 < \varepsilon \text{ and } \frac{x_1^3(x_2^2 + \widehat{p}_1 + x_1)}{\widehat{p}_2} < 0 \\
u &= -\frac{x_2^2 + \widehat{p}_1 + x_1}{\widehat{p}_2}\ .
\end{aligned} \tag{124}$$

□

Proposition (106) generalizes results established by Sastry and Isidori [29] for systems that are input-output linearizable via state feedback, and by Taylor et al. [31] for state-feedback linearizable systems:

Corollary [Sastry and Isidori [29, Relative Degree One]] **(125)**
Let a and A in equation (S_p) be affine in u, let the system to be controlled have a single input, i.e., $m = 1$ and, finally, let Π be an open subset of $\mathbb{R}^l$ which satisfies assumption ICS. Assume the existence of two C^2 functions $h : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that:

1. *$h(\mathcal{E}) = 0$,*
2. *the functions $\frac{\partial h}{\partial x}(x)a(x,u)$ and $\frac{\partial h}{\partial x}(x)A(x,u)$ are* known,
3. *we have for all (x, p) in $\mathbb{R}^n \times \Pi$ (with notation (13)):*

$$\frac{\partial h}{\partial x}(x)\ (b(x) + B(x)p)\ \neq\ 0\,, \tag{126}$$

4. *$(h(x), \varphi(x))$ is a diffeomorphism and defines new coordinates with which the system (S_{p^*}) can be rewritten as:*

$$\dot{h} = \frac{\partial h}{\partial x}(x)\ [a(x,u) + A(x,u)p^*] \tag{127}$$

$$\dot{\varphi} = Z(\varphi, h)\,, \tag{128}$$

where Z is a function which is further assumed to be globally Lipschitz.

Assume also that $\varphi(\mathcal{E})$ is an exponentially stable equilibrium point in $\mathbb{R}^{n-1}$ of:

$$\dot{\varphi}\ =\ Z(\varphi, 0)\,. \tag{129}$$

Under these conditions, we can find functions V, u_{n} and $\mathcal{P}$ such that the corresponding dynamic controller (105) solves the Adaptive Stabilization problem.

Assumption (126) means that each system (S_p), with $p \in \Pi$, is of relative degree one with respect to the output function h (see Isidori [6]). And the exponential stability of (129) implies that (S_{p^*}) is globally exponentially minimum phase.

Proof. Assumption Λ-LP is met with the choice:

$$\Lambda(x,t) \;=\; \frac{\partial h}{\partial x}(x)\,. \tag{130}$$

Then, let:

$$\Omega \;=\; \Omega_0 \;=\; \mathbb{R}^n\,, \qquad \alpha_0 \;=\; +\infty\,, \tag{131}$$

and choose the function V independent of p as simply:

$$V(x) \;=\; h(x)^2\,. \tag{132}$$

According to [29, Proposition 2.1], assumption BO is satisfied. An argument similar to the one used in Example (59) (replace K by ε in [29, 2.27]) proves that assumption CO holds also.

Now from (126), there exists a C^1 function $u_{\mathrm{n}} : \mathbb{R}^n \times \Pi \to \mathbb{R}$ such that:

$$\frac{\partial h}{\partial x}(x)\; [a(x, u_{\mathrm{n}}(x,p) + A(x, u_{\mathrm{n}}(x,p))p] \;=\; -c\, h(x)\,, \tag{133}$$

with c a strictly positive constant. With (127), this implies that assumption PRS holds.

In conclusion, the controller (105) may be employed, with the function $\mathcal{P}$ given by assumption ICS and $\alpha_1 = +\infty$. □

Corollary [Taylor et al. [31]] (134)
Let, in equation (S_p), the functions a and A be known *and affine in u, let Π be an open subset of $\mathbb{R}^l$ which satisfies assumption ICS and, finally, let p_0 be a* known *vector in Π. Assume there exist an open neighborhood Ω of $\mathcal{E}$ in $\mathbb{R}^n$ and three* known *functions:*

$$\begin{array}{lll} \Phi \;: \Omega \to \mathbb{R}^n & \textit{of class } C^2 \textit{ which is a diffeomorphism,} \\ w_1 : \Omega \to \mathbb{R}^m & \textit{of class } C^1\,, \textit{ and} \\ w_2 : \Omega \to \mathcal{GL}(m,\mathbb{R}) & \textit{of class } C^1\,, \end{array}$$

such that:

1. *by letting:*

$$\varphi \;=\; \Phi(x) \qquad \textit{and} \qquad u \;=\; w_1(x) + w_2(x)\,\vartheta\,, \tag{135}$$

the time derivative of φ along the solutions of (S_{p_0}) satisfies, for all ϑ in $\mathbb{R}^m$:

$$\dot{\varphi} \;=\; C\,\varphi \;+\; D\,\vartheta\,, \tag{136}$$

where D is an $n \times m$ matrix and C is an $n \times n$ matrix satisfying:

$$P\,C \;+\; C^{\mathrm{T}} P \;=\; -I\,, \tag{137}$$

with P a symmetric positive definite matrix,

2. *for all (x,p,u) in $\Omega \times \Pi \times \mathbb{R}^m$, we have, with notation (13):*

$$\operatorname{rank}\{b(x) + B(x)p\} \;=\; m \tag{138}$$

$$[A_0(x) \;+\; u \odot B(x)]\, p :\in\; \operatorname{span}\{b(x) + B(x)p_0\}\;. \tag{139}$$

Under these conditions, we can find functions V, u_{n} and $\mathcal{P}$ and a constant α_1 such that the corresponding dynamic controller (105) solves the Adaptive Stabilization problem.

Note that (136) simply implies that $u = w_1 + w_2\vartheta$ is a feedback linearizing control for the system (S_{p_0}) in the coordinates $\varphi = \Phi(x)$.

Proof. Under the above assumptions, it is proved in [31, Proposition S] that:

$$\Phi^{-1}(0) \,=\, \mathcal{E} \tag{140}$$

and there exists a known C^1 function $u_{\mathrm{n}} \,:\, \Omega \times \Pi \,\rightarrow\, \mathbb{R}^m$ such that, for all (x,p) in $\Omega \times \Pi$, we have:

$$\frac{\partial \Phi}{\partial x}(x)\,(a(x,p) + A(x,p)u_{\mathrm{n}}(x,p)) \,=\, C\,\Phi(x)\,. \tag{141}$$

It follows that assumption PRS is satisfied if we choose:

$$V(x) \,=\, \Phi(x)^{\mathrm{T}} P \Phi(x)\,. \tag{142}$$

To check that assumption BO is satisfied, let us define α_0 as the largest (possibly infinite) real number such that the set:

$$\left\{\varphi \,\middle|\, \varphi^{\mathrm{T}} P \varphi \,<\, \alpha_0\right\}$$

is contained in $\Phi(\Omega)$. Since $\Phi(\mathcal{E}) = 0$, Φ is a diffeomorphism, and P is a positive definite matrix, the so-defined α_0 is strictly positive. Then the sets:

$$\Gamma_\alpha \,=\, \Phi^{-1}\left\{\varphi \,\middle|\, \varphi^{\mathrm{T}} P \varphi \,\leq\, \alpha\right\} \tag{143}$$

are compact subsets of Ω for all $\alpha < \alpha_0$. It follows that:

$$V(x) \,\leq\, \alpha \quad \Longrightarrow \quad x \,\in\, \Gamma_\alpha\,, \tag{144}$$

which implies that BO is satisfied with $\Omega_0 = \Omega$.

Assumption CO holds since:

$$V(x(t)) \,\rightarrow\, 0 \,\Longrightarrow\, \Phi(x(t)) \,\rightarrow\, 0 \,\Longrightarrow\, x(t) = \Phi^{-1}(\Phi(x(t))) \,\rightarrow\, \mathcal{E}\,. \tag{145}$$

Since assumption Λ-LP is also satisfied with the function Λ equal to the $n \times n$ identity matrix, the controller (105) may be employed, with the function $\mathcal{P}$ given by assumption ICS and any α_1 such that $0 < \alpha_1 \leq \alpha_0$. □

One of the features of the result established by Taylor et al. [31] is that it gives a sufficient condition which guarantees the existence of a function V not depending on p while satisfying assumptions BO, PRS and CO. This condition is assumption (139), called the strict matching condition in [31, Assumption L]. It turns out that (with notation (13)), when $B \equiv 0$ and $b(\mathcal{E})$ has full rank m, this condition is also the necessary and sufficient condition for the existence of a regular static state-feedback law:

$$u \,=\, c(x) \,+\, d(x)\,w \,+\, e(x)p\,, \tag{146}$$

which decouples the state vector x from p seen as a measured disturbance (see [6, Proposition 5.5.1 and Proposition 7.3.1]). As pointed out by Pomet in [19, Sect. 1.2.3], it follows that, by applying the control:

$$u \,=\, e(x)\,(p - p_0) \,+\, w \tag{147}$$

to any system (S_p) (with input u), $p \in \Pi$, we obtain the particular system (S_{p_0}) (with input w). In fact the same result holds if $B \not\equiv 0$:

Lemma [19, Théorème 2.2] (148)
Let, in equation (S_p), the functions a and A be known *and affine in u, let $\varPi$ be an open subset of $\mathbb{R}^l$, let $\varOmega$ be an open neighborhood of $\mathcal{E}$ in $\mathbb{R}^n$ and, finally, let p_0 be a* known *vector in $\varPi$. Assume:*

$$\operatorname{rank}\{b(x) + B(x)p\} \;=\; m \qquad \forall\,(x,p) \,\in\, \varOmega \times \varPi\,. \tag{149}$$

Under these conditions, assumption (139), i.e.,

$$[A_0(x) \,+\, u \odot B(x)]\,p :\in\; \operatorname{span}\{b(x) + B(x)p_0\} \qquad \forall\,(x,p,u) \,\in\, \varOmega \times \varPi \times \mathbb{R}^m\,, \tag{150}$$

is equivalent to the following proposition:
There exist two known *C^1 functions:*

$$c \,:\, \varOmega \times \varPi \,\to\, \mathbb{R}^m\,, \qquad d \,:\, \varOmega \times \varPi \,\to\, \mathcal{GL}(m,\mathbb{R})\,,$$

such that, for all (x,p,w) in $\varOmega \times \varPi \times \mathbb{R}^m$:

$$a(x,w) \,+\, A(x,w)\,p_0 \,=\, a\,(x\,,\, c(x,p) + d(x,p)w) \,+\, A\,(x\,,\, c(x,p) + d(x,p)w)\,p\,. \tag{151}$$

A straightforward consequence is:

Corollary [19, Proposition 8.13] (152)
Let, in equation (S_p), the functions a and A be known *and affine in u, let $\varPi$ be an open subset of $\mathbb{R}^l$ which satisfies assumption ICS and, finally, let p_0 be a* known *vector in $\varPi$. Assume there exists an open neighborhood $\varOmega$ of $\mathcal{E}$ in $\mathbb{R}^n$ such that:*

1. $\operatorname{rank}\{b(x) + B(x)p\} \;=\; m \qquad \forall\,(x,p) \,\in\, \varOmega \times \varPi$,
2. $[A_0(x) + u \odot B(x)]\,p \,\in\, \operatorname{span}\{b(x) + B(x)p_0\} \qquad \forall\,(x,p,u) \,\in\, \varOmega \times \varPi \times \mathbb{R}^m$,
3. *there exist two* known *functions:*

$$V \,:\, \varOmega \,\to\, \mathbb{R}_+ \text{ of class } C^2\,, \qquad u_0 \,:\, \varOmega \,\to\, \mathbb{R}^m \text{ of class } C^1\,,$$

such that the functions $\frac{\partial V}{\partial x}(x)\,a(x,u)$ and $\frac{\partial V}{\partial x}(x)\,A(x,u)$ are known*, assumptions BO and CO hold and we have, for all x in $\varOmega$:*

$$\frac{\partial V}{\partial x}(x)\,[a(x,u_0(x)) + A(x,u_0(x))\,p_0] \,\leq\, 0\,, \tag{153}$$

where the inequality is strict iff $V(x) \neq 0$.

Under these conditions, we can find functions V, u_{n} and $\mathcal{P}$ and a constant α_1 such that the corresponding dynamic controller (105) solves the Adaptive Stabilization problem.

Proof. Assumption $\varLambda$-LP holds with:

$$\varLambda(x,t) \;=\; \frac{\partial V}{\partial x}(x)\,. \tag{154}$$

Then, since assumptions BO, CO and ICS are satisfied, it is sufficient to define a function u_{n} meeting assumption PRS. According to Lemma (148), let:

$$u_{\mathrm{n}}(x,p) \;=\; c(x,p) \,+\, d(x,p)\,u_0(x)\,. \tag{155}$$

This is a C^1 function, and inequality (47) in assumption PRS is satisfied since (151) and (153) hold. □

Assumption (139) is very restrictive. Fortunately, (139) is not necessary for the existence of a function V independent of p, as illustrated below.

Example: **(156)**
Consider the following system:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 + p^* (x_3 + x_2)(x_1 + 2x_2 + 2x_3) . \end{aligned} \tag{157}$$

We have:

$$\begin{aligned} a_0(x) \;=\; \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}, \qquad A_0(x) = \begin{pmatrix} 0 \\ 0 \\ (x_3+x_2)(x_1+2x_2+2x_3) \end{pmatrix}, \\ b(x) \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad B(x) \;=\; 0 . \end{aligned} \tag{158}$$

Hence, even though b is of full rank 1, assumption (139) does not hold. Nevertheless, assumptions BO, PRS and CO are satisfied if we choose V independent of p as:

$$V(x_1, x_2, x_3) \;=\; \frac{x_3^2}{2} \;+\; \frac{(x_3+x_2)^2}{2} \;+\; \frac{(x_1+2x_2+2x_3)^2}{2} \tag{159}$$

and u_{n} as:

$$u_{\mathrm{n}}(x_1, x_2, x_3, p) = -3x_3 - 5x_2 - 3x_1 - p\,(x_3+x_2)(6x_3+5x_2+2x_1) . \tag{160}$$

Indeed, V is positive definite and radially unbounded and a straightforward computation leads to the following expression for the time derivative of V along the solutions of (157):

$$\begin{aligned} \dot{V} = &\; - x_3^2 - (x_3+x_2)^2 \\ &+ (x_1+2x_2+2x_3)\,[u + 2x_1 + 3x_2 + x_3 + p\,(x_3+x_2)(6x_3+5x_2+2x_1)] . \end{aligned} \tag{161}$$

□

2.2 Case: V Dependent on p with a "Matching Condition"

When V depends on p, choosing $\dot{\widehat{p}}$ as in (87) is not sufficient to guarantee that $\dot{W}$ is negative when W is given by (85). This follows from the disturbing term $\frac{\partial V}{\partial p}(x, \widehat{p})\dot{\widehat{p}}$ present in (86). To overcome this difficulty, a very fruitful idea proposed by Kanellakopoulos, Kokotovic and Marino [8] (see also Middleton and Goodwin [15]) is to compensate this measurable disturbing term by modifying the control to:

$$u \;=\; u_{\mathrm{n}}(x, \widehat{p}) \;+\; v . \tag{162}$$

Indeed, in this case, (86) becomes:

$$\begin{aligned} \dot{W} \leq &\left[-\frac{\partial V}{\partial x}(x, \widehat{p})\, A(x, u_{\mathrm{n}}(x, \widehat{p}) + v) + \dot{\widehat{p}}^{\mathrm{T}} \right] [\widehat{p} - p^*] + \frac{\partial V}{\partial p}(x, \widehat{p})\dot{\widehat{p}} \\ &+ \frac{\partial V}{\partial x}(x, \widehat{p})\, [a(x, u_{\mathrm{n}}(x, \widehat{p}) + v) - a(x, u_{\mathrm{n}}(x, \widehat{p})) \\ &+ (A(x, u_{\mathrm{n}}(x, \widehat{p}) + v) - A(x, u_{\mathrm{n}}(x, \widehat{p})))\, \widehat{p}] . \end{aligned} \tag{163}$$

Hence, taking:

$$\dot{\widehat{p}} = \left[\frac{\partial V}{\partial x}(x,\widehat{p})\, A(x,u_{\rm n}(x,\widehat{p})+v)\right]^{\rm T}, \qquad \widehat{p}(0) \in \Pi\,, \tag{164}$$

and choosing v as a solution of:

$$\begin{aligned} 0 = \frac{\partial V}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} + \frac{\partial V}{\partial x}(x,\widehat{p})\,&[a(x,u_{\rm n}(x,\widehat{p})+v) - a(x,u_{\rm n}(x,\widehat{p})) \\ &+ (A(x,u_{\rm n}(x,\widehat{p})+v) - A(x,u_{\rm n}(x,\widehat{p})))\,\widehat{p}]\,, \end{aligned} \tag{165}$$

we guarantee that $\dot{W}$ is negative. However, a new difficulty may arise from the fact that (164) and (165) is a system of implicit equations to be solved in $\dot{\widehat{p}}$ and v. A solution smoothly depending on x and $\widehat{p}$ may not exist. Pomet [19] (see also [21]) has proposed the following way to avoid this difficulty, when a and A are affine in u:
With the control u as in (162), i.e.,

$$u = u_{\rm n}(x,p) + v\,, \tag{166}$$

we may embed the family of systems (S_p) into the following larger family $(S_{p,q})$ (with notation (13)):

$$\dot{x} = a_0 + (u_{\rm n}+v)\odot b + (A_0 + u_{\rm n}\odot B)\,p + v\odot B\,q\,, \quad (p,q) \in \Pi\times\Pi\,. \tag{$S_{p,q}$}$$

Then, we modify the function W to:

$$W(x,\widehat{p}) = V(x,\widehat{p}) + \frac{1}{2}\,\|\widehat{p}-p^*\|^2 + \frac{1}{2}\,\|\widehat{q}-p^*\|^2\,. \tag{167}$$

As above, the time derivative of this function W is made negative if we choose:

$$\begin{aligned} \dot{\widehat{p}} &= \left[\frac{\partial V}{\partial x}(x,\widehat{p})\,(A_0(x)+u_{\rm n}(x,\widehat{p})\odot B(x))\right]^{\rm T}, \qquad \widehat{p}(0) \in \Pi \\ \dot{\widehat{q}} &= \left[\frac{\partial V}{\partial x}(x,\widehat{p})\,v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right)\odot B(x)\right]^{\rm T}, \qquad \widehat{q}(0) \in \Pi \\ u &= u_{\rm n}(x,\widehat{p}) + v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right). \end{aligned} \tag{168}$$

This dynamic controller is well-defined with unique solutions if we assume:

Assumption MC (Matching Condition) (169)
The functions a and A are affine in u and there exists a known C^1 *function $v(x,p,q,\partial)$ from $\Omega\times\Pi\times\Pi\times\mathbb{R}^l$ to $\mathbb{R}^m$ satisfying:*

$$\frac{\partial V}{\partial p}(x,p)\,\partial + \frac{\partial V}{\partial x}(x,p)\,v\odot[b(x)+B(x)\,q] = 0\,. \tag{170}$$

We remark that, if V is independent of p, this assumption is trivially satisfied (by $v=0$). Also, the way this assumption is stated is too restrictive for our purpose. In order to make $\dot{W}$ negative it is sufficient that (170) be satisfied for the particular case $\partial = \dot{\widehat{p}}$ and not for all $\partial \in \mathbb{R}^l$. We have stated assumption MC is these terms to allow us independent choices for the function V and for $\dot{\widehat{p}}$. Indeed, relaxing (170) by replacing ∂ by $\dot{\widehat{p}}$ implies that V, $u_{\rm n}$ and $\dot{\widehat{p}}$ must be designed all together so that BO, CO, PRS and MC are satisfied at the same time.

Example: **(171)**
Consider the system, with $n = 2$ and $l = 1$:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_2^2 \left(x_1 + p\, x_2^2\right). \end{aligned} \tag{172}$$

Assumptions BO, PRS and CO are satisfied, with:

$$V(x_1, x_2, p) = x_2^2 \,+\, (x_1 + x_2 + p\, x_2^2)^2 \tag{173}$$

$$u_{\rm n}(x_1, x_2, p) = -x_2 \,-\, x_2^3 \,-\, (x_1 + p\, x_2^2)\left(1 + x_2^2\,(1 + 2\, p\, x_2)\right). \tag{174}$$

Assumption Λ-LP is also satisfied with:

$$\Lambda(x_1, x_2, t) = \frac{\partial V}{\partial x}(x_1, x_2, \widehat{p}(t)). \tag{175}$$

Finally, assumption MC holds since in this case equation (170) is:

$$2\, x_2^2\,(x_1 + x_2 + p\, x_2^2)\, \partial \,+\, 2\,(x_1 + x_2 + p\, x_2^2)\, v = 0, \tag{176}$$

which is satisfied by the choice

$$v \,=\, -x_2^2\, \partial. \tag{177}$$

□

Finally, to guarantee that $\widehat{p}(t)$ and $\widehat{q}(t)$ remain in Π and that $V(x(t), \widehat{p}(t))$ remains smaller than α_0, we have to modify further the controller (168) as in the previous section:

$$\begin{aligned} \dot{\widehat{p}} &= \operatorname{Proj}\left(I, \widehat{p}\,, \frac{{\alpha_1}^2}{(V(x,\widehat{p}) - \alpha_1)^2} \left[\frac{\partial V}{\partial x}(x,\widehat{p})\,(A_0(x) + u_{\rm n}(x,\widehat{p}) \odot B(x))\right]^{\rm T}\right) \\ \dot{\widehat{q}} &= \operatorname{Proj}\left(I, \widehat{q}\,, \frac{{\alpha_1}^2}{(V(x,\widehat{p}) - \alpha_1)^2} \left[\frac{\partial V}{\partial x}(x,\widehat{p})\, v(x, \widehat{p}, \widehat{q}, \dot{\widehat{p}}) \odot B(x)\right]^{\rm T}\right) \\ u &= u_{\rm n}(x, \widehat{p}) \,+\, v\left(x, \widehat{p}, \widehat{q}, \dot{\widehat{p}}\right), \end{aligned} \tag{178}$$

with $\widehat{p}(0)$ and $\widehat{q}(0)$ in Π_1. We have:

Proposition **(179)**
Let assumptions BO, PRS, ICS and MC hold. Assume also that assumption Λ-LP is satisfied with:

$$\Lambda(x, t) \,=\, \frac{\partial V}{\partial x}(x, \widehat{p}(t)). \tag{180}$$

If, in (178), α_1 is chosen smaller than or equal to α_0, then all the solutions $(x(t), \widehat{p}(t), \widehat{q}(t))$ of (S_{p^*})-(178), with $x(0)$ in Ω_0 and $V(x(0), \widehat{p}(0)) < \alpha_1$, are well-defined on $[0, +\infty)$, unique, bounded and:

$$\lim_{t \to \infty} V(x(t), \widehat{p}(t)) \,=\, 0. \tag{181}$$

It follows that the Adaptive Stabilization problem is solved if assumption CO holds also.

Proof. The system we consider is:

$$\begin{aligned}
\dot{x} &= a\left(x\,,\, u_{\rm n}(x,\widehat{p})\,+\,v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right) + A\left(x\,,\, u_{\rm n}(x,\widehat{p})\,+\,v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right)\,p^*\\
\dot{\widehat{p}} &= \operatorname{Proj}\left(I,\widehat{p}\,,\,\frac{{\alpha_1}^2}{(V(x,\widehat{p})-\alpha_1)^2}\left[\frac{\partial V}{\partial x}(x,\widehat{p})\,(A_0(x)+u_{\rm n}(x,\widehat{p})\odot B(x))\right]^{\rm T}\right)\\
\dot{\widehat{q}} &= \operatorname{Proj}\left(I,\widehat{q}\,,\,\frac{{\alpha_1}^2}{(V(x,\widehat{p})-\alpha_1)^2}\left[\frac{\partial V}{\partial x}(x,\widehat{p})\,v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\odot B(x)\right]^{\rm T}\right),
\end{aligned} \tag{182}$$

with $\widehat{p}(0)$ and $\widehat{q}(0)$ in Π_1. ¿From our smoothness assumptions on the functions a, A, $u_{\rm n}$ and V and with Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{p},\widehat{q}) \in\ \Omega\times\Pi\times\Pi \qquad \text{and} \qquad V(x,\widehat{p})\ <\ \alpha_1\,. \tag{183}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{p}(t),\widehat{q}(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite, and satisfying (183) for all t in $[0,T)$. Applying Point 5 of Lemma (103), we also know that $\widehat{p}(t)\in\Pi_1$ and $\widehat{q}(t)\in\Pi_1$ for all t in $[0,T)$. Then, let us compute the time derivative, along such a solution, of the function W defined by:

$$W(x,\widehat{p})\ =\ \frac{\alpha_1 V(x,\widehat{p})}{\alpha_1 - V(x,\widehat{p})}\ +\ \frac{1}{2}\,\|\widehat{p}-p^*\|^2\ +\ \frac{1}{2}\,\|\widehat{q}-p^*\|^2\ . \tag{184}$$

With assumption PRS, Point 4 of Lemma (103) and equation (170) with $\partial=\dot{\widehat{p}}$ satisfied by $v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right)$, we get:

$$\begin{aligned}
\dot{W} &\leq 0\ \text{ if } V(x(t),\widehat{p}(t))=0\\
&< 0\ \text{ if } V(x(t),\widehat{p}(t))\neq 0\,.
\end{aligned} \tag{185}$$

¿From there, we conclude exactly as in the proof of Proposition (106). □

Example: **(186)**
The assumptions of Proposition (179) are satisfied by the system (172) of Example (171). For this system, the Adaptive Stabilization problem is solved by the following dynamic controller (using (164), (173) and (177)):

$$\begin{aligned}
\dot{\widehat{p}} &= 2\,\left[x_2\,+\,\left(x_1+x_2+\widehat{p}x_2^2\right)(1+2\widehat{p}x_2)\right]\,x_2^4\\
u &= -x_2\,-\,x_2^3\,-\,(x_1+p\,x_2^2)\left(1+x_2^2\,(1+2\,p\,x_2)\right)\\
&\quad -2\,x_2^2\,\left[x_2\,+\,\left(x_1+x_2+\widehat{p}x_2^2\right)(1+2\widehat{p}x_2)\right]\,x_2^4\,.
\end{aligned} \tag{187}$$

□

Proposition (179) generalizes to the case where V is not radially unbounded a result established by Kanellakopoulos, Kokotovic and Marino for state-feedback linearizable systems [8]:

Corollary [Kanellakopoulos, Kokotovic and Marino [8]] (188)
Let, in equation (S_p), the functions a and A be known *and affine in u, let p_0 be a* known *vector in $\mathbb{R}^l$. Assume there exist a bounded open subset Ω of $\mathbb{R}^n$, an open neighborhood $\mho$ of p_0 in $\mathbb{R}^l$ and three* known *functions:*

$$\begin{array}{lll} \Psi\ : \Omega \to \mathbb{R}^n & \text{of class } C^2 \text{ which is a diffeomorphism,} \\ w_1 : \Omega \to \mathbb{R}^m & \text{of class } C^1\,, \text{ and} \\ w_2 : \Omega \to \mathcal{GL}(m,\mathbb{R}) & \text{of class } C^1\,, \end{array}$$

such that:

1. *by letting:*
$$\varphi = \Psi(x) \qquad \text{and} \qquad u = w_1(x) + w_2(x)\,\vartheta\,, \tag{189}$$
the time derivative of φ along the solutions of (S_{p_0}) satisfies, for all ϑ in $\mathbb{R}^m$:
$$\dot{\varphi} = C\,\varphi + D\,\vartheta\,, \tag{190}$$
where D is an $n \times m$ matrix and C is an $n \times n$ matrix satisfying:
$$P\,C + C^{\mathrm{T}}P = -I\,, \tag{191}$$
with P a symmetric positive definite matrix,
2. *for all (x,p,u) in $\Omega \times \mho \times \mathbb{R}^m$, we have, with notation (13):*
$$\operatorname{rank}\{b(x) + B(x)p\} = m \tag{192}$$
$$\begin{array}{c} u \odot B(x)\,p \in \operatorname{span}\{b(x) + B(x)p_0\} \\ A_0(x)\,p :\in \operatorname{span}\{b(x) + B(x)p_0\,,\ [a_0 + A_0p_0\,, \operatorname{span}\{b + Bp_0\}](x)\}\,, \end{array} \tag{193}$$
where $[\cdot\,,\,\cdot]$ denotes the Lie bracket.

Under these conditions and if p^* is close enough to p_0, we can find functions V, u_{n} and $\mathcal{P}$ and a constant α_1 such that the corresponding dynamic controller (178) solves the Adaptive Stabilization problem.

Proof. ¿From the sections "State diffeomorphism" and "Ideal feedback control" in [8] (see also [9]), there exist an open neighborhood Π_d of p_0 and three known functions:

$$\begin{array}{lll} \Phi\ : \Omega \times \Pi_d \to \mathbb{R}^n & \text{of class } C^2\text{, a diffeomorphism for each } p, \\ u_{\mathrm{n}} : \Omega \times \Pi_d \to \mathbb{R}^m & \text{of class } C^1\,, \text{ and} \\ v\ : \Omega \times \Pi_d \times \Pi_d \times \mathbb{R}^l \to \mathbb{R}^m & \text{of class } C^1\,, \end{array}$$

such that $\mathcal{E}$ belongs to Ω,

$$\Phi(\mathcal{E},p^*) = 0\,, \quad \Phi(x,p_0) = \Psi(x) \quad \forall\, x \in \Omega\,, \tag{194}$$

and, for all (x,p,q,∂) in $\Omega \times \Pi_d \times \Pi_d \times \mathbb{R}^l$, we have:

$$\frac{\partial\Phi}{\partial x}(x,p)\,[a(x,p) + A(x,p)u_{\mathrm{n}}(x,p)] = C\,\Phi(x,p) \tag{195}$$

$$\frac{\partial\Phi}{\partial p}(x,p)\,\partial + \frac{\partial\Phi}{\partial x}(x,p)\,v \odot [b(x) + B(x)\,q] = 0\,. \tag{196}$$

Then, let us choose the function V as:

$$V(x,p) = \Phi(x,p)^{\mathrm{T}} P\,\Phi(x,p)\,. \tag{197}$$

In order to choose the set Π and the scalar α_0, let us define a function F by:

$$\begin{array}{rcl} F : \Omega \times \Pi_d & \to & \mathbb{R}^n \times \Pi_d \,. \\ (x,p) & & (\Phi(x,p)\,,\,p) \end{array} \tag{198}$$

This function is a diffeomorphism satisfying $F(\Psi^{-1}(0), p_0) = (0, p_0)$. Since $\Omega \times \Pi_d$ is an open neighborhood of $(\Psi^{-1}(0), p_0)$ and p^* is assumed to be close enough from p_0, there exist strictly positive real numbers α_0 and π such that:

$$\|p^* - p_0\|^2 \,<\, \frac{\pi}{1+\varepsilon}\;, \tag{199}$$

with $0 < \varepsilon < 1$, and the set:

$$\left\{ (\varphi, p) \;\middle|\; \|p - p_0\|^2 \,<\, \pi \quad \text{and} \quad \varphi^{\mathrm{T}} P\, \varphi \,<\, \alpha_0 \right\}$$

is contained in $F\,(\Omega \times \Pi_d)$ and contains $\mathcal{E}$. Then, let us define the function $\mathcal{P}$ by:

$$\mathcal{P}(p) \;=\; \frac{2}{\varepsilon}\left[\frac{1+\varepsilon}{\pi}\,\|p - p_0\|^2 - 1\right], \tag{200}$$

and the set Π by:

$$\Pi \;=\; \{p \mid \mathcal{P}(p) \,<\, 1+\varepsilon\}\;. \tag{201}$$

Assumption ICS is satisfied.

¿From (195) and (196), it is clear that assumptions PRS and MC are satisfied. To check that assumption BO is satisfied, we remark that for all compact subsets $\mathcal{K}$ of Π and all $\alpha < \alpha_0$, the sets:

$$\left\{ (x,p) \;\middle|\; \Phi(x,p)^{\mathrm{T}} P\, \Phi(x,p) \,\le\, \alpha \quad \text{and} \quad p \,\in\, \mathcal{K} \right\}$$

are compact subsets of $\Omega \times \Pi$ and therefore their projections:

$$\Gamma_{\alpha,\mathcal{K}} \;=\; \left\{ x \;\middle|\; \exists\, p \,\in\, \mathcal{K} \,:\, \Phi(x,p)^{\mathrm{T}} P\, \Phi(x,p) \,\le\, \alpha \right\} \tag{202}$$

are compact subsets of Ω. It follows that:

$$V(x,p) \,\le\, \alpha \quad \text{and} \quad p \,\in\, \mathcal{K} \qquad \Longrightarrow \qquad x \,\in\, \Gamma_{\alpha,\mathcal{K}}\,, \tag{203}$$

which implies that BO is satisfied with $\Omega_0 = \Omega$.

The proof that assumption CO holds follows from the remark:

$$V(x(t), \widehat{p}(t)) \,\to\, 0 \quad \Longrightarrow \quad \Phi(x(t, \widehat{p}(t))) \,\to\, 0 \,=\, \Phi(\mathcal{E}, p^*) \tag{204}$$

and the proof of [9, Theorem 1] (see also Example (68)).

Since assumption Λ-LP is also satisfied with the function Λ equal to the $n \times n$ identity matrix, the controller (178) applies. □

One of the nice results proved by Kanellakopoulos, Kokotovic and Marino is that for feedback linearizable systems, assumption (193), called the extended matching condition in [8, Assumption E] is necessary and sufficient for assumption MC to hold. In fact, as for the case where V does not depend on p, this assumption (193) relies on the fact that we can transform any system (S_p) into a particular one (S_{p_0}) but this time by using both feedback and diffeomorphism. Precisely, we have:

Lemma [19, Théorème 2.9 and Lemme 8.14] (205)
Let the functions a and A in equation (S_p) be affine in u and let $\mathcal{U}$ be an open neighborhood of $\mathcal{E}$ in $\mathbb{R}^n$. Assume that, in $\mathcal{U}$ (with notation (13)):

1. *The distribution* $\operatorname{span}\{b + Bp^*\}$ *is involutive with constant full rank m,*
2. *The distribution* $\operatorname{span}\{b + Bp^*, [a_0 + A_0 p^*, \operatorname{span}\{b + Bp^*\}]\}$ *has constant rank.*

Under these conditions, assumption (193), i.e.,

$$\begin{aligned} u \odot B(x)\, p :\in \ & \operatorname{span}\{b(x) + B(x) p_0\} \\ A_0(x)\, p :\in \ & \operatorname{span}\{b(x) + B(x) p_0\, , \, [a_0 + A_0 p_0\, , \operatorname{span}\{b + B p_0\}]\,(x)\}\ , \end{aligned} \tag{206}$$

is equivalent to the following proposition:
There exist an open neighborhood Ω of $\mathcal{E}$, an open neighborhood Π of p^* of $\mathbb{R}^l$, a vector p_0 in Π and four smooth functions:

$$\begin{array}{lll} \Phi : \Omega \times \Pi \ \to \ \mathbb{R}^n & & \textit{of class } C^2, \textit{ a diffeomorphism for each } p, \\ c \ : \Omega \times \Pi \ \to \ \mathbb{R}^m & & \textit{of class } C^1\,, \\ d \ : \Omega \times \Pi \ \to \ \mathcal{GL}(m, \mathbb{R}) & & \textit{of class } C^1\,, \textit{ and} \\ v \ : \Omega \times \Pi \times \Pi \times \mathbb{R}^l \ \to \ \mathbb{R}^m & & \textit{of class } C^1\,, \end{array}$$

such that, for all (x, p, q, ∂) in $\Omega \times \Pi \times \Pi \times \mathbb{R}^l$, we have:

$$\frac{\partial \Phi}{\partial p}(x,p)\, \partial \, + \, \frac{\partial \Phi}{\partial x}(x,p)\, v \odot [b(x) + B(x)\, q] \ = \ 0\,, \tag{207}$$

and, for all (x, p, w) in $\Omega \times \Pi \times \mathbb{R}^m$, we have:

$$\begin{aligned} & a(\Phi(x,p), w) \ + \ A(\Phi(x,p), w)\, p_0 \\ & \qquad = \ \frac{\partial \Phi}{\partial x}(x,p)\, [a\,(x\, , \, c(x,p) + d(x,p) w) \ + \ A\,(x\, , \, c(x,p) + d(x,p) w)\, p]\,. \end{aligned} \tag{208}$$

Comparing (207) and (170) in assumption MC, we understand the importance of this result. It gives us a possible route for finding functions V and u_{n} satisfying all our assumptions BO, PRS, CO and MC. Indeed, if

1. the conditions of Lemma (205) are satisfied, and
2. for the particular system (S_{p_0}), with p_0 given by Lemma (205), there exist V_0 and u_0 satisfying assumptions BO, PRS and CO but this time with the family (S_p) reduced to the single element (S_{p_0}), i.e., $p = p^* = p_0$,

then, by choosing:

$$\begin{aligned} V(x,p) \ &= \ V_0(\Phi(x,p)) \\ u_{\mathrm{n}}(x,p) \ &= \ c(x,p) \ + \ d(x,p)\, u_0\, (\Phi(x,p))\,, \end{aligned} \tag{209}$$

assumptions BO, PRS, CO and MC are necessarily satisfied.

However, it is important to notice that finding a solution v in $\mathbb{R}^m$ to the n equations (207) is more difficult than finding a solution v in $\mathbb{R}^m$ to the single equation (170). For single-input two-dimensional affine in u systems with $b(\mathcal{E}) \neq 0$ and $B(x) \in \operatorname{span}\{b(x)\}$, the assumptions of Lemma (205) are "generically" satisfied. Indeed, we can expect that, for almost all x close to $\mathcal{E}$, we have:

$$\operatorname{span}\{b(x)\, , \, [a_0\, , \, b]\,(x)\} \ = \ \mathbb{R}^2\,. \tag{210}$$

However, the set of such x's may not be a neighborhood of $\mathcal{E}$, implying that (207) may not hold. Nevertheless, d'Andréa-Novel, Pomet and Praly [1] have shown that, for this two-dimensional case, there are explicit expressions for the functions V, u_{n} and v satisfying assumptions BO, PRS and MC.

Compared with Proposition (76), Proposition (179) states that, when p^* is unknown, the solution to the Adaptive Stabilization problem given by the Lyapunov design requires the Matching Condition (MC). And nothing is known if this condition does not hold.

Another but very particular possibility to handle the case where V depends on p is to choose:

$$W \;=\; \frac{\alpha_1 V(x,p^*)}{\alpha_1 - V(x,p^*)} \;+\; \frac{1}{2}\,\|\widehat{p}-p^*\|^2 \qquad \text{and} \qquad u \;=\; u_{\mathrm{n}}(x,\widehat{p})\,. \tag{211}$$

This gives:

$$\dot{W} = \dot{\widehat{p}}^{\mathrm{T}}\,[\widehat{p}-p^*] + \frac{{\alpha_1}^2}{(\alpha_1 - V(x,p^*))^2}\,\frac{\partial V}{\partial x}(x,p^*)\,[a(x,u_{\mathrm{n}}(x,\widehat{p})) + A(x,u_{\mathrm{n}}(x,\widehat{p}))p^*]\,. \tag{212}$$

Then, using assumption PRS, we get:

$$\begin{aligned}\dot{W} \;\leq\; & \dot{\widehat{p}}^{\mathrm{T}}\,[\widehat{p}-p^*] \;+\; \frac{{\alpha_1}^2}{(\alpha_1 - V(x,p^*))^2}\,\frac{\partial V}{\partial x}(x,p^*)\\ & \times [a(x,u_{\mathrm{n}}(x,\widehat{p})) - a(x,u_{\mathrm{n}}(x,p^*)) + (A(x,u_{\mathrm{n}}(x,\widehat{p})) - A(x,u_{\mathrm{n}}(x,p^*)))\,p^*]\,.\end{aligned} \tag{213}$$

There is no general expression of $\dot{\widehat{p}}$ not depending on p^* and making the right-hand side of this inequality negative. However, Slotine and Li [30] have shown that in the particular case of rigid robot arms, it is possible to find functions V and u_{n} which satisfy assumption BO, PRS and CO and are such that:

There exists a C^1 function $Z \,:\, \Omega\times\Pi \;\rightarrow\; \mathbb{R}^l$ such that, for the particular value p^* but for all $(x,p) \;\in\; \Omega\times\Pi$, we have:

$$\begin{aligned}[p^*-p]^{\mathrm{T}}\,Z(x,p) \;=\; & \frac{{\alpha_1}^2}{(\alpha_1 - V(x,p^*))^2}\,\frac{\partial V}{\partial x}(x,p^*)\\ & \times [a(x,u_{\mathrm{n}}(x,p)) - a(x,u_{\mathrm{n}}(x,p^*)) + (A(x,u_{\mathrm{n}}(x,p)) - A(x,u_{\mathrm{n}}(x,p^*)))\,p^*]\,.\end{aligned} \tag{214}$$

Indeed, with such a property, the Adaptive Stabilization problem is solved by choosing:

$$\dot{\widehat{p}} \;=\; \mathrm{Proj}\,(I\,,\,\widehat{p}\,,\,Z(x,\widehat{p}))\,. \tag{215}$$

3 Estimation Design

Another way to obtain dynamic controllers to solve the Adaptive Stabilization problem is to hope that a separation principle holds, i.e., that we can get an estimate $\widehat{p}$ of p^* by using a parameter estimator and simultaneously we apply the control u_{n} with the unknown parameter vector p^* replaced by its estimate $\widehat{p}$, i.e.,

$$u \;=\; u_{\mathrm{n}}(x,\widehat{p})\,. \tag{216}$$

For the estimation, we note that, thanks to assumption Λ-LP, equation (S_p) is linear in p, i.e., it can be rewritten as:

$$z = Z\,p\,, \tag{217}$$

with:

$$z = \dot{x} - a(x,u) \qquad \text{and} \qquad Z = A(x,u)\,. \tag{218}$$

Also, the vector p^* we want to estimate is constant, i.e., it is a solution of:

$$\dot{p} = 0\,. \tag{219}$$

Estimating p^* is then equivalent to observing, through the observation equation (217), the state vector p which obeys the (trivial) dynamic equation (219). From linear observer theory [7], we know that an observer can be written as:

$$\dot{\widehat{p}} = -K\left(Z\,\widehat{p} - z\right)\,, \tag{220}$$

with K the observer gain. Unfortunately, such an observer cannot be implemented, since it makes use of the unknown quantities $\dot{x}$, $a(x,u)$ and $A(x,u)$. This difficulty is handled as follows:

About the unmeasured time derivative $\dot{x}$, it is quite clear that an integration should help us. It turns out that two ways of implementing this integration are fruitful:

1. equation error filtering, and
2. regressor filtering.

These techniques will be presented hereafter.

To deal with the fact that only the functions $\Lambda\,a$ and $\Lambda\,A$ are known, we select an integer k and a C^2 function $h : \Omega \times \Pi \to \mathbb{R}^k$ such that the functions $\frac{\partial h}{\partial x}(x,p)\,a(x,u)$ and $\frac{\partial h}{\partial x}(x,p)\,A(x,u)$ are known functions of (x,u,p), i.e., assumption Λ-LP is met with:

$$\Lambda(x,t) = \frac{\partial h}{\partial x}(x,\widehat{p}(t))\,. \tag{221}$$

This function h is called the *observation function*. For any C^1 time functions $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$, the time derivative of $h(x(t),\widehat{p}(t))$ with $x(t)$ a solution of (S_{p^*}) satisfies:

$$\dot{h} = \frac{\partial h}{\partial x}(x,\widehat{p})\left(a(x,u) + A(x,u)\,p^*\right) + \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}}\,. \tag{222}$$

This is again an equation linear in p^* and equality (217) is again satisfied but with the following definitions of z and Z:

$$\begin{aligned} z &= \dot{h} - \frac{\partial h}{\partial x}(x,\widehat{p})\,a(x,u) - \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \\ Z &= \frac{\partial h}{\partial x}(x,\widehat{p})\,A(x,u)\,. \end{aligned} \tag{223}$$

Example: System (17) Continued **(224)**
For the system (17) rewritten as (50) in Example (49), let us choose the observation function h as:

$$h(x_1, x_2, p_1, p_2) \;=\; \frac{1}{2}\, x_1^2\,. \tag{225}$$

We get:

$$\dot{\overline{h(x_1(t), x_2(t), p_1(t), p_2(t))}} \;=\; x_1^3(t)\, \left(x_2^2(t) + p_1^* + p_2^*\, u(t)\right). \tag{226}$$

By letting:

$$\begin{aligned} z(t) &= \dot{\overline{h(x_1(t), x_2(t), p_1(t), p_2(t))}} \;-\; x_1^3(t)\, x_2^2(t) \\ Z(t) &= \begin{pmatrix} x_1^3(t) & x_1^3(t)\, u(t) \end{pmatrix}, \end{aligned} \tag{227}$$

we obtain equation (217), i.e.,

$$z(t) \;=\; Z(t) \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix}. \tag{228}$$

□

3.1 Equation Error Filtering

The estimator is obtained from the observer (220) where the so called equation error $Z\widehat{p} - z$ is replaced by a filtered version. Namely, let e in $\mathbb{R}^k$ be defined as follows:

$$\dot{e} \,+\, r(e, x, \widehat{p})\, e \;=\; Z\, \widehat{p} \,-\, z\,, \tag{229}$$

or equivalently, using (223):

$$\begin{aligned} \dot{\widehat{h}} &= -r(e, x, \widehat{p})\, e \,+\, \frac{\partial h}{\partial x}(x, \widehat{p})\, A(x, u)\, \widehat{p} \,+\, \frac{\partial h}{\partial x}(x, \widehat{p})\, a(x, u) \,+\, \frac{\partial h}{\partial p}(x, \widehat{p})\, \dot{\widehat{p}} \\ e &= \widehat{h} \,-\, h(x, \widehat{p})\,, \end{aligned} \tag{230}$$

with r a positive C^1 function defined in $\mathbb{R}^k \times \Omega \times \Pi$. Definitely, e can be obtained without knowing $\dot{h}$ and the estimate $\widehat{p}$ is then given by:

$$\dot{\widehat{p}} \;=\; \mathrm{Proj}\,(I\,,\, \widehat{p}\,,\, -K\, e)\,, \qquad \widehat{p}(0) \;\in\; \Pi_1\,, \tag{231}$$

where the matrix M used in the function Proj is the identity matrix and, typically:

$$K \;=\; Z^{\mathrm{T}} \;=\; \left(\frac{\partial h}{\partial x}(x, \widehat{p})\, A(x, u)\right)^{\mathrm{T}}. \tag{232}$$

Unfortunately, as in the Lyapunov design case, if, instead of (216), we implement the control:

$$u \;=\; u_{\mathrm{n}}(x, \widehat{p}) \,+\, v\,, \tag{233}$$

where v depends on $\dot{\widehat{p}}$, we have again an implicit definition of $\dot{\widehat{p}}$ when v is explicitly involved in the right-hand side of (231). In such a case, if the functions a and A are affine in u, we derive the estimator from equation $(S_{p,q})$ instead of equation (S_p). Using notation (13), this leads to the following observer:

$$\begin{aligned}
\dot{\widehat{h}} &= -r(e,x,\widehat{p})\,e + \frac{\partial h}{\partial x}(x,\widehat{p})\,[a_0(x) + A_0(x)\widehat{p} + u_{\rm n}(x,\widehat{p}) \odot (b(x) + B(x)\widehat{p})] \\
&\quad + \frac{\partial h}{\partial x}(x,\widehat{p})\,v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot (b(x) + B(x)\widehat{q}) + \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \\
e &= \widehat{h} - h(x,\widehat{p}) \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(I,\widehat{p}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\,(A_0(x) + u_{\rm n}(x,\widehat{p}) \odot B(x))\right]^{\rm T} e\right)\,,\ \widehat{p}(0) \in \Pi_1 \\
\dot{\widehat{q}} &= \mathrm{Proj}\left(I,\widehat{q}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\,v(x,\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot B(x)\right]^{\rm T} e\right)\,,\ \widehat{q}(0) \in \Pi_1\,.
\end{aligned} \tag{234}$$

In the sequel, such a modified estimator will be implicitly assumed to be used whenever the Matching Condition holds.

We have:

Lemma **(235)**

Assume ICS is satisfied. For any C^1 time function $u : \mathbb{R}_+ \to \mathbb{R}^m$, all the solutions $(x(t),\widehat{p}(t),\widehat{h}(t))$ of (S_{p^*})-(230)-(231)-(232) defined on $[0,T)$ with $x(t)$ remaining in Ω satisfy for all t in $[0,T)$:

1. $\widehat{p}(t) \in \Pi_1$
2. $\|\widehat{p}(t) - p^*\|^2 + \|e(t)\|^2 + 2\int_0^t r\,\|e\|^2 \le \|\widehat{p}(0) - p^*\|^2 + \|e(0)\|^2$.

Similarly for the solutions of (S_{p^*,p^*})-(234), we have, for all t in $[0,T)$:

3. $\widehat{p}(t) \in \Pi_1$ *and* $\widehat{q}(t) \in \Pi_1$
4. $\left\|\begin{matrix}\widehat{p}(t) - p^* \\ \widehat{q}(t) - p^*\end{matrix}\right\|^2 + \|e(t)\|^2 + 2\int_0^t r\,\|e\|^2 \le \left\|\begin{matrix}\widehat{p}(0) - p^* \\ \widehat{q}(0) - p^*\end{matrix}\right\|^2 + \|e(0)\|^2$.

Proof. Point 1 is a straightforward consequence of Point 5 of Lemma (103). For Point 2, let us denote η the input of the dynamic system (229), i.e.,

$$\eta = Z\,\widehat{p} - z\,. \tag{236}$$

This system with output e is passive, namely, it satisfies for all t in $[0,T)$:

$$\int_0^t e^{\rm T}\eta = \int_0^t e^{\rm T}\left(\dot{e} + re\right) \tag{237}$$

$$= \frac{1}{2}\,\|e(t)\|^2 - \frac{1}{2}\,\|e(0)\|^2 + \int_0^t r\,\|e\|^2 \ge -\frac{1}{2}\,\|e(0)\|^2\,. \tag{238}$$

On the other hand, the dynamic system (231) with input e and output y defined by:

$$y = Z\,(p^* - \widehat{p}) \tag{239}$$

is also passive. Indeed, we have thanks to Point 4 of Lemma (103):

$$\int_0^t y^{\mathrm{T}} e = \int_0^t (\widehat{p} - p^*)^{\mathrm{T}} (-Z^{\mathrm{T}}) e \tag{240}$$

$$\geq \int_0^t (\widehat{p} - p^*)^{\mathrm{T}} \operatorname{Proj}\left(I\,,\, \widehat{p}\,,\, -Z^{\mathrm{T}}\, e\right) \tag{241}$$

$$\geq \int_0^t (\widehat{p} - p^*)^{\mathrm{T}} \dot{\widehat{p}} \tag{242}$$

$$\geq \frac{1}{2} \left\|\widehat{p}(t) - p^*\right\|^2 - \frac{1}{2} \left\|\widehat{p}(0) - p^*\right\|^2 \tag{243}$$

$$\geq -\frac{1}{2} \left\|\widehat{p}(0) - p^*\right\|^2 . \tag{244}$$

Noting that p^* satisfies:

$$z \;=\; Z\, p^* , \tag{245}$$

the two passive systems are interconnected with:

$$y = -\eta\, . \tag{246}$$

Then, Point 2 follows directly from standard passivity theorems (see Landau [12]) or more directly by comparing (238) and (243). The proof of Points 3 and 4 is similar. □

In fact, as emphasized by Landau [12] and expected from this proof, a similar Lemma would be obtained if, instead of the filter (229), we would have used any passive operator. In particular, as already noticed for adaptive control of linear systems by Narendra and Valavani [17], Pomet [19] has shown that, by choosing the identity matrix for the observation function h and a copy of the controlled system itself as the filter (229), we can rederive the adaptation law (178) of the Lyapunov design. Indeed, assume that BO, PRS and MC hold and define e as the output of, instead of (229), the following system with input η and state χ, with notation (13):

$$\begin{aligned} \dot{\chi} &= a_0(\chi) + A_0(\chi)\widehat{p} + u_{\mathrm{n}}(\chi, \widehat{p}) \odot (b(\chi) + B(\chi)\widehat{p}) \\ &\qquad + v\left(\chi, \widehat{p}, \widehat{q}, \dot{\widehat{p}}\right) \odot (b(\chi) + B(\chi)\widehat{q}) \;+\; \eta \\ e &= \frac{\partial V}{\partial \chi}(\chi, \widehat{p})^{\mathrm{T}} . \end{aligned} \tag{247}$$

To see that this system is passive, we look at the time derivative of $V(\chi(t), \widehat{p}(t))$. We get:

$$\dot{V} \;=\; \frac{\partial V}{\partial \chi}\left[a_0 + A_0\, \widehat{p} + u_{\mathrm{n}} \odot (b + B\, \widehat{p})\right] \;+\; \frac{\partial V}{\partial \chi}\, v \odot (b + B\widehat{q}) \;+\; \frac{\partial V}{\partial \widehat{p}} \dot{\widehat{p}} \;+\; \frac{\partial V}{\partial \chi}\, \eta\, . \tag{248}$$

Hence, the definition of e, (47) in assumption PRS and (170) in assumption MC give readily:

$$\dot{V} \;\leq\; e^{\mathrm{T}} \eta\, . \tag{249}$$

Therefore, for any solution of (247) defined on $[0, T)$, we have:

$$\int_0^t e^{\mathrm{T}} \eta \;\geq\; V(\chi(t), \widehat{p}(t)) \;-\; V(\chi(0), \widehat{p}(0)) \qquad \forall\, t \;\in\; [0, T)\, . \tag{250}$$

In the proof of Lemma (235), we have seen that if equation $(S_{p,q})$ is satisfied with $p = p^*$ and $q = p^*$, the following system with input e, state $\begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix}$ and ouput y:

$$\begin{aligned}
\dot{\widehat{p}} &= \mathrm{Proj}\left(I, \widehat{p}\,, -\left[(A_0(\mathcal{X}) + u_\mathrm{n}(\mathcal{X},\widehat{p}) \odot B(\mathcal{X}))\right]^\mathrm{T} e\right)\,,\ \widehat{p}(0) \in \Pi_1 \\
\dot{\widehat{q}} &= \mathrm{Proj}\left(I, \widehat{q}\,, -\left[v(\mathcal{X},\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot B(\mathcal{X})\right]^\mathrm{T} e\right)\,,\ \widehat{q}(0) \in \Pi_1 \\
y &= (A_0(\mathcal{X}) + u_\mathrm{n}(\mathcal{X},\widehat{p}) \odot B(\mathcal{X}))\,(\widehat{p} - p^*)\ +\ v\left(\mathcal{X},\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot B(\mathcal{X})\,(\widehat{q} - p^*)
\end{aligned} \tag{251}$$

is passive, namely:

$$\int_0^t e^\mathrm{T} y\ \geq\ \frac{1}{2}\left\|\begin{matrix}\widehat{p}(t) - p^* \\ \widehat{q}(t) - p^*\end{matrix}\right\|^2 - \frac{1}{2}\left\|\begin{matrix}\widehat{p}(0) - p^* \\ \widehat{q}(0) - p^*\end{matrix}\right\|^2 \qquad \forall\, t\ \in\ [0,T)\,. \tag{252}$$

It follows that the observer defined by equations (247) and (251) is such that $V(\mathcal{X}(t),\widehat{p}(t))$, $\widehat{p}(t)$ and $\widehat{q}(t)$ are bounded if η is chosen such that:

$$\begin{aligned}
\eta &= -y \\
&= (A_0(\mathcal{X}) + u_\mathrm{n}(\mathcal{X},\widehat{p}) \odot B(\mathcal{X}))\,(p^* - \widehat{p}) \\
&\qquad\qquad + v\left(\mathcal{X},\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot B(\mathcal{X})\,(p^* - \widehat{q})\,.
\end{aligned} \tag{253}$$

It remains to check that the observer (247)–(251) can be implemented while satisfying this equality, i.e., with (247) rewritten as:

$$\begin{aligned}
\dot{\mathcal{X}} = a_0(\mathcal{X}) + A_0(\mathcal{X})p^* + u_\mathrm{n}(\mathcal{X},\widehat{p}) \odot (b(\mathcal{X}) + B(\mathcal{X})p^*) \\
+ v\left(\mathcal{X},\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot (b(\mathcal{X}) + B(\mathcal{X})p^*)\ +\ \eta\,.
\end{aligned} \tag{254}$$

The right-hand side of this equation is nothing but a copy of the system (S_{p^*}) with the control:

$$u\ =\ u_\mathrm{n}(x,\widehat{p})\ +\ v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right). \tag{255}$$

It follows that if, in (247), we choose the initial condition

$$\mathcal{X}(0)\ =\ x(0)\,, \tag{256}$$

then, necessarily, for all t:

$$\mathcal{X}(t)\ =\ x(t)\,. \tag{257}$$

This implies that, in fact, in the observer, the $\dot{\mathcal{X}}$ equation does not need to be implemented, but that we simply have to replace $\mathcal{X}$ by x in the definition of e, $\dot{\widehat{p}}$ and $\dot{\widehat{q}}$. This gives exactly the adaptation law in (178) provided by the Lyapunov design.

3.2 Regressor Filtering

To overcome the difficulty of having $\dot{x}$ or more precisely $\dot{h}$ unmeasured, another way to implement an integration follows from the following remark:
Let z_{f} and Z_{f} be the following filtered quantities:

$$\begin{aligned} \dot{z}_{\mathrm{f}} + \rho(x,\widehat{p},u)\, z_{\mathrm{f}} = z \quad , \quad z_{\mathrm{f}}(0) = 0 \\ \dot{Z}_{\mathrm{f}} + \rho(x,\widehat{p},u)\, Z_{\mathrm{f}} = Z \quad , \quad Z_{\mathrm{f}}(0) = 0 \,, \end{aligned} \tag{258}$$

or equivalently, using (223):

$$\begin{aligned} \dot{\widehat{z}}_{\mathrm{f}} &= \rho(x,\widehat{p},u)\, z_{\mathrm{f}} + \frac{\partial h}{\partial x}(x,\widehat{p})\, a(x,u) + \frac{\partial h}{\partial p}(x,\widehat{p})\, \dot{\widehat{p}} \ , \ \widehat{z}_{\mathrm{f}}(0) = h(x(0),\widehat{p}(0)) \\ z_{\mathrm{f}} &= h(x,\widehat{p}) - \widehat{z}_{\mathrm{f}} \\ \dot{Z}_{\mathrm{f}} &= -\rho(x,\widehat{p},u)\, Z_{\mathrm{f}} + \frac{\partial h}{\partial x}(x,\widehat{p})\, A(x,u) \qquad\qquad , \ Z_{\mathrm{f}}(0) = 0 \,, \end{aligned} \tag{259}$$

where ρ is a C^1 positive function. Clearly, the knowledge of $\dot{h}$ is not needed and equation (S_{p^*}) gives:

$$z_{\mathrm{f}} = Z_{\mathrm{f}}\, p^* \,. \tag{260}$$

This equation is again linear in p^*. This yields the following linear observer for p^*:

$$\begin{aligned} e &= Z_{\mathrm{f}}\, \widehat{p} - z_{\mathrm{f}} \\ \dot{\widehat{p}} &= \mathrm{Proj}\,(M\,,\, \widehat{p}\,,\, -K\, e)\,, \qquad \widehat{p}(0) \in \Pi_1\,, \end{aligned} \tag{261}$$

where, typically, the observer gain K and the matrix M, used in the function Proj, are given by:

$$\begin{aligned} K &= M\, Z_{\mathrm{f}}^{\mathrm{T}} r(x,\widehat{p},u,e) \\ \dot{M} &\geq -(2-\varepsilon_1)\, M\, Z_{\mathrm{f}}^{\mathrm{T}} Z_{\mathrm{f}}\, M\, r(x,\widehat{p},u,e)\,, \quad \varepsilon_2\, M \leq I\,, \quad I \leq \varepsilon_3\, M(0)\,, \end{aligned} \tag{262}$$

where $0 < \varepsilon_1 < 2$, $0 < \varepsilon_3$, $0 < \varepsilon_2$, r is a strictly positive C^1 function and, if the observer gain K is allowed to decay to zero with time:

$$\dot{M} \leq -\varepsilon_4\, M\, Z_{\mathrm{f}}^{\mathrm{T}} Z_{\mathrm{f}}\, M\, r(x,\widehat{p},u,e)\,, \quad 2-\varepsilon_1 \geq \varepsilon_4 > 0\,. \tag{263}$$

Note that, in (261), $\dot{\widehat{p}}$ depends on u via the dependence of the observer gain K on r. Unfortunately, in this case, it is useless to extend the parameterization by embedding (S_p) into $(S_{p,q})$, since r will remain a factor in the extended observer gain. It follows that to make sure that $\dot{\widehat{p}}$ is well-defined, we shall impose that r does not depend on u whenever u is allowed to depend on $\dot{\widehat{p}}$.

We have:

Lemma: (264)
Assume ICS is satisfied. There exists a positive continuous function k_1 such that, for any C^1 time function $u : \mathbb{R}_+ \to \mathbb{R}^m$, all $(x(t), \widehat{p}(t), \widehat{z}_{\mathrm{f}}(t), Z_{\mathrm{f}}(t), M(t))$, solutions of (S_{p^*})-(259)-(261)-(262), defined on $[0,T)$ with $(x(t), M(t))$ remaining in $\Omega \times \mathcal{M}$ satisfy for all t in $[0,T)$:

1. $\widehat{p}(t) \in \Pi_1$
2. $\varepsilon_2 \|\widehat{p}(t) - p^*\|^2 + \varepsilon_1 \int_0^t r\, \|e\|^2 \leq \varepsilon_3 \|\widehat{p}(0) - p^*\|^2$
3. *if (263) holds, then* $\int_0^t \left\|\dot{\widehat{p}}\right\| \leq k_1(p^*, \widehat{p}(0))$.

Moreover, for all constant k_2, the property:

$$\frac{\|Z(t)\|}{\rho(x(t), \widehat{p}(t), u(t))} \leq k_2 \quad \forall\, t \in [0,T) \tag{265}$$

implies the property:

$$\|Z_{\mathrm{f}}(t)\| \leq k_2 \quad \text{and} \quad \|e(t)\| \leq k_2 \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\, \|\widehat{p}(0) - p^*\| \quad \forall\, t \in [0,T). \tag{266}$$

Note that a trivial way of meeting the condition (265) is to choose:

$$\rho(x, \widehat{p}, u) = 1 + \|Z\|. \tag{267}$$

Proof. Point 1 is a straightforward consequence of Point 5 of Lemma (103). For Point 2, first note that (260) and (261) imply:

$$e = Z_{\mathrm{f}}\,(\widehat{p} - p^*). \tag{268}$$

Then, let us consider the time derivative of:

$$W_1(t) = \frac{1}{2}\,(\widehat{p}(t) - p^*)^{\mathrm{T}} M^{-1}(t)\,(\widehat{p}(t) - p^*) \tag{269}$$

along the solutions of (261). This is a well-defined time function since by assumption $M(t) \in \mathcal{M}$. First note that, from (262), we get:

$$\dot{\overline{M^{-1}}} \leq (2 - \varepsilon_1)\, Z_{\mathrm{f}}^{\mathrm{T}} Z_{\mathrm{f}}\, r, \tag{270}$$

then, with Point 4 of Lemma (103) and (268), we get successively:

$$\begin{aligned}
\dot{W}_1 &= (\widehat{p} - p^*)^{\mathrm{T}} \left[\frac{1}{2}\dot{\overline{M^{-1}}}\,(\widehat{p} - p^*) + M^{-1}\dot{\widehat{p}}\right] \\
&\leq \left(1 - \frac{\varepsilon_1}{2}\right) r\, \|Z_{\mathrm{f}}\,(\widehat{p} - p^*)\|^2 \\
&\qquad + (\widehat{p} - p^*)^{\mathrm{T}} M^{-1} \operatorname{Proj}\left(M, \widehat{p}, -r\, M\, Z_{\mathrm{f}}^{\mathrm{T}} e\right) \\
&\leq \left(1 - \frac{\varepsilon_1}{2}\right) r\, \|Z_{\mathrm{f}}\,(\widehat{p} - p^*)\|^2 - r\,(\widehat{p} - p^*)^{\mathrm{T}} Z_{\mathrm{f}}^{\mathrm{T}} e \\
&\qquad + (\widehat{p} - p^*)^{\mathrm{T}} M^{-1} \operatorname{Proj}\left(M, \widehat{p}, -r\, M\, Z_{\mathrm{f}}^{\mathrm{T}} e\right) - (\widehat{p} - p^*)^{\mathrm{T}} \left(-r Z_{\mathrm{f}}^{\mathrm{T}} e\right) \\
&\leq \left(1 - \frac{\varepsilon_1}{2}\right) r\, \|Z_{\mathrm{f}}\,(\widehat{p} - p^*)\|^2 - r\,(\widehat{p} - p^*)^{\mathrm{T}} Z_{\mathrm{f}}^{\mathrm{T}} e \\
&\leq -\frac{\varepsilon_1}{2}\, r\, \|e\|^2.
\end{aligned} \tag{271}$$

On the other hand, from (262), we have:

$$\frac{\varepsilon_2}{2}\left\|\widehat{p}(t)-p^*\right\|^2 \,\leq\, W_1(t) \,\leq\, \frac{\lambda_{\max}\left\{M^{-1}(t)\right\}}{2}\left\|\widehat{p}(t)-p^*\right\|^2 , \tag{272}$$

where $\lambda_{\max}\{\cdot\}$ denotes the maximum eigenvalue. Point 2 follows readily and also with (271):

$$\begin{aligned}\int_0^t (\widehat{p}-p^*)^{\mathrm{T}}\left(-r\,Z_{\mathrm{f}}^{\mathrm{T}} e\right) - \int_0^t (\widehat{p}-p^*)^{\mathrm{T}} M^{-1}\,\mathrm{Proj}\left(M\,,\,\widehat{p}\,,\,-rM\,Z_{\mathrm{f}}^{\mathrm{T}} e\right)&\\ \leq\,\frac{\varepsilon_3}{2}\,\left\|\widehat{p}(0)-p^*\right\|^2 .&\end{aligned} \tag{273}$$

To prove Point 3, let us define χ as follows:

$$\begin{aligned}\chi = 0 &\quad \text{if}\ \mathcal{P}(\widehat{p})\,\leq\,0\quad\text{or}\quad \tfrac{\partial\mathcal{P}}{\partial p}(\widehat{p})\,M\,Z_{\mathrm{f}}^{\mathrm{T}} e\,\geq\,0\\ = \mathcal{P}(\widehat{p}) &\ \text{if}\ \mathcal{P}(\widehat{p})\,>\,0\ \ \text{and}\ \ \tfrac{\partial\mathcal{P}}{\partial p}(\widehat{p})\,M\,Z_{\mathrm{f}}^{\mathrm{T}} e\,<\,0\,.\end{aligned} \tag{274}$$

With the definition (102) of Proj and Point 3 in Lemma (103), we get:

$$(\widehat{p}-p^*)^{\mathrm{T}}\left(-Z_{\mathrm{f}}^{\mathrm{T}} e\right) - (\widehat{p}-p^*)^{\mathrm{T}} M^{-1}\mathrm{Proj}\left(M\,,\,\widehat{p}\,,\,-M\,Z_{\mathrm{f}}^{\mathrm{T}} e\right)$$

$$= \chi\,\frac{\left(-\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right)\left(\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\,(\widehat{p}-p^*)\right)}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}} \tag{275}$$

$$= \chi\,\frac{\left|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right|\left(\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\,(\widehat{p}-p^*)\right)}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}} \tag{276}$$

$$\geq\, D^*\,\chi\,\frac{\left|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right|\left\|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\right\|}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}}\ . \tag{277}$$

Then, the expression of $\dot{\widehat{p}}$ in (261) and the inequality on $\dot{M}$ in (263) give with (262):

$$\left\|\dot{\widehat{p}}\right\| = \left\|\mathrm{Proj}\left(M\,,\,\widehat{p}\,,\,-r\,M\,Z_{\mathrm{f}}^{\mathrm{T}} e\right)\right\| \tag{278}$$

$$\leq r\left\|MZ_{\mathrm{f}}^{\mathrm{T}} e\right\| \,+\, r\,\chi\,\frac{\left|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right|\left\|M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\right\|}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}}\ . \tag{279}$$

And, with the Cauchy Schwarz inequality and (262), we get:

$$\begin{aligned}\int_0^t \left\|\dot{\widehat{p}}\right\| \leq &\sqrt{\int_0^t r\left\|e\right\|^2}\sqrt{\int_0^t \mathrm{tr}\left\{rMZ_{\mathrm{f}}^{\mathrm{T}} Z_{\mathrm{f}} M\right\}}\\ &\qquad + \int_0^t \frac{r\chi}{\varepsilon_2}\,\frac{\left|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right|\left\|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\right\|}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}}\\ \leq &\sqrt{\int_0^t r\left\|e\right\|^2}\sqrt{\int_0^t \mathrm{tr}\left\{\frac{-\dot{M}}{\varepsilon_4}\right\}}\\ &\qquad + \int_0^t \frac{r\chi}{\varepsilon_2}\,\frac{\left|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})MZ_{\mathrm{f}}^{\mathrm{T}} e\right|\left\|\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})\right\|}{\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})M\frac{\partial\mathcal{P}}{\partial p}(\widehat{p})^{\mathrm{T}}}\ ,\end{aligned} \tag{280}$$

where tr$\{\cdot\}$ denotes the trace of a matrix. The proof of Point 3 is now straightforward. We use the inequalities given by Point 2, (273) and (277) and get:

$$\int_0^t \left\|\dot{\widehat{p}}\right\| \leq \sqrt{\frac{\varepsilon_3}{\varepsilon_2}} \, \|\widehat{p}(0) - p^*\| \, \sqrt{\frac{\mathrm{tr}\,\{M(0)\}}{\varepsilon_4}} + \frac{\varepsilon_3}{2\,D^*\,\varepsilon_2} \, \|\widehat{p}(0) - p^*\|^2 \, . \tag{281}$$

Finally, we prove the implication stated in the Lemma. To simplify the notations, let:

$$\rho(t) \,=\, \rho\left(x(t), \widehat{p}(t), u(t)\right). \tag{282}$$

The solution $Z_{\mathrm{f}}(t)$ of (258) is for all t in $[0, T)$:

$$Z_f(t) \,=\, \int_0^t \exp\left(-\int_s^t \rho(\tau)\, d\tau\right) Z(s)\, ds\,. \tag{283}$$

Therefore, if for all t:

$$\|Z(t)\| \,\leq\, k_2\, \rho(t)\,, \tag{284}$$

we have:

$$\|Z_{\mathrm{f}}(t)\| \leq \int_s^t \exp\left(-\int_s^t \rho(\tau)\, d\tau\right) \|Z(s)\|\, ds \tag{285}$$

$$\leq k_2 \int_0^t \exp\left(-\int_0^t \rho(\tau)\, d\tau\right) \rho(s)\, ds \tag{286}$$

$$\leq k_2\,. \tag{287}$$

Finally, the inequality on $\|e\|$ is a straightforward consequence of this inequality, Point 2 and (268). □

An important drawback of this result is that it requires a particular initial condition for the initial condition $\widehat{z_{\mathrm{f}}}(0)$. Choosing another initial condition may create difficulties if at the same time r is not guaranteed to be bounded.

3.3 Estimation Design

The estimation design of a dynamic controller for solving the Adaptive Stabilization problem consists in choosing the observation function h and either the equation error filtering or the regressor filtering technique. This provides an estimate $\widehat{p}$ which we use in the nominal control u_{n} to obtain the control:

$$u \,=\, u_{\mathrm{n}}(x, \widehat{p}) \,+\, v\,, \tag{288}$$

where, as in the Lyapunov design case, v may be introduced to counteract the effects of updating $\widehat{p}$, so that v may depend on $\dot{\widehat{p}}$. In both the equation error filtering case and the regressor filtering case, the estimator is trying to find a function $\widehat{p}$ fitting as well as possible the $\dot{h}$ equation (222). On the other hand, to show that we are solving the Adaptive Stabilization problem, we see, from the proof of Proposition (76), that both the estimation and the control should be such that $\dot{V}$ is negative. With assumption PRS, such an objective is met by the control u_{n} as long as the $\dot{V}$ equation is well-fitted by $\widehat{p}$. Consequently, the fact that the Adaptive Stabilization problem is solved depends crucially on the fact that, when the $\dot{h}$ equation is well-fitted, the same holds for the

$\dot{V}$ equation. More precisely, this fact relies on the properties of the following set value maps:

$$\begin{aligned} F(p,\eta) &= \{(p,\nu) \mid \nu = V(x,p) \quad \text{and} \quad h(x,p) = \eta\} \\ F^{\dagger}(p,\nu) &= \{(p,\eta) \mid \eta = h(x,p) \quad \text{and} \quad V(x,p) = \nu\} \,. \end{aligned} \tag{289}$$

4 Estimation Design with the Observation Function $h = \frac{\alpha_1 V}{\alpha_1 - V}$

When we choose the observation function h as:

$$h(x,p) = \frac{\alpha_1 V(x,p)}{\alpha_1 - V(x,p)} \,, \tag{290}$$

the above set value maps F and $F^{\dagger}$ are in fact standard applications defined by:

$$F(p,\eta) = \left(p\,,\, \frac{\eta\,\alpha_1}{\eta + \alpha_1}\right) \qquad \text{and} \qquad F^{\dagger}(p,\nu) = \left(p\,,\, \frac{\nu\,\alpha_1}{\alpha_1 - \nu}\right). \tag{291}$$

As we will show this is a very favorable situation compared with other possible choices for the observation function h.

4.1 Equation Error Filtering

Let $\widehat{p}$ be obtained from the equation error filtering technique (230)–(232). If we implement the control:

$$u = u_{\mathrm{n}}(x,\widehat{p}) \,, \tag{292}$$

then, with (47) in assumption PRS, equation (230) gives:

$$\dot{\widehat{h}} \leq -r(e,x,\widehat{p})\,e + \frac{{\alpha_1}^2}{(V(x,\widehat{p}) - \alpha_1)^2}\,\frac{\partial V}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \,, \tag{293}$$

where:

$$\widehat{h} = \frac{\alpha_1 V}{\alpha_1 - V} + e \,. \tag{294}$$

If, instead, assumption MC holds, let $\widehat{p}$ and $\widehat{q}$ be obtained from (234) and use the control:

$$u = u_{\mathrm{n}}(x,\widehat{p}) + v\left(x, \widehat{p}, \widehat{q}, \dot{\widehat{p}}\right) , \tag{295}$$

with v given by (170) in MC with $\partial = \dot{\widehat{p}}$. Then, inequality (293) reduces to:

$$\dot{\widehat{h}} \leq -r(e,x,\widehat{p})\,e \,. \tag{296}$$

Since V is positive and e is bounded (see Lemma (235)), $\widehat{h}$ is lower bounded. To conclude that $\widehat{h}$ is also upper bounded and therefore V is strictly smaller than α_1, we would need to know that re is in L^1 but we only know, from Lemma (235), that $\sqrt{r}\,e$ is in L^2. In fact to prove boundedness of $\widehat{h}$, we need to strengthen assumption PRS:

Assumption URS (Uniform Reduced-order Stabilizability) (297)
There exists a positive constant c such that, for all (x,p) *in* $\Omega \times \Pi$*, we have:*

$$\frac{\partial V}{\partial x}(x,p)\left[a(x,u_{\mathrm{n}}(x,p)) + A(x,u_{\mathrm{n}}(x,p))\,p\right] \le -c\,V(x,p)\,. \tag{298}$$

Example: System (17) Continued (299)
Consider the system (17) rewritten as (50). Assumptions BO, CO and URS are satisfied if we choose:

$$u_{\mathrm{n}}(x_1,x_2,p_1,p_2) = -\frac{x_2^2+p_1+x_1}{p_2}\,, \tag{300}$$

and:

$$\begin{aligned} V(x_1,x_2,p_1,p_2) = V(x_1) &= \left(x_1+\frac{2}{3}\right)^2 && \text{if} \quad x_1 \le -1 \\ &= \frac{1}{9}\exp\left(3\left(1-\frac{1}{x_1^2}\right)\right) && \text{if} \quad -1 < x_1 < 1 \\ &= \left(x_1-\frac{2}{3}\right)^2 && \text{if} \quad 1 \le x_1\,. \end{aligned} \tag{301}$$

Indeed:

1. The function V is of class C^2 with:

$$\begin{aligned} \frac{d^2V}{dx_1{}^2}(x_1) &= 2 && \text{if} \quad |x_1| \ge 1 \\ &= 2\,\frac{2-x_1^2}{x_1^6}\,\exp\left(3\left(1-\frac{1}{x_1^2}\right)\right) && \text{if} \quad |x_1| < 1\,. \end{aligned} \tag{302}$$

2. If $V(x_1)$ is bounded, so is x_1, and if $V(x_1)$ tends to 0, so does x_1. Hence, as in Example (49), BO and CO are satisfied.
3. Finally, assumption URS is met, since we get:

$$\begin{aligned} &\frac{\partial V}{\partial x}(x,p)\left[a(x,u_{\mathrm{n}}(x,p)) + A(x,u_{\mathrm{n}}(x,p))\,p\right] \\ &\qquad = -2\left(x_1+\frac{2}{3}\right)x_1^3 && \text{if} \quad x_1 \le -1 \\ &\qquad\quad -\frac{2}{3}\exp\left(3\left(1-\frac{1}{x_1^2}\right)\right) && \text{if} \quad -1 < x_1 < 1 \\ &\qquad\quad -2\left(x_1-\frac{2}{3}\right)x_1^3 && \text{if} \quad 1 \le x_1 \\ &\qquad \le -2\,V(x,p)\,. \end{aligned} \tag{303}$$

□

Example: System (25) Continued (304)
For the system (25), assume the function L satisfies the following growth condition: there exist a positive constant γ and an integer j such that:

$$|L(y)| \le \gamma\left(|y| + |y|^j\right). \tag{305}$$

Under this condition, we can find functions V and u_{n} such that assumptions BO, CO and URS are satisfied by the non-minimal state-space representation (27) of (25). Indeed, let $\Phi(p)$ be the following invertible matrix:

$$\Phi(p) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & p & 0 \\ 0 & 0 & 0 & 1 & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{306}$$

It allows us to define new coordinates:

$$\chi \;=\; \Phi(p)\, x \;=\; \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} , \tag{307}$$

and to rewrite (27) in the following block form:

$$\dot{\chi} \;=\; \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \chi \;+\; \begin{pmatrix} G_1 \\ 0 \end{pmatrix} [u + L(y)] \;+\; \begin{pmatrix} 0 \\ G_2 \end{pmatrix} L(y) \tag{308}$$

$$y \;=\; (H_1 \;\, 0)\chi + \delta(t)\,, \tag{309}$$

where the pair $(F_1\,,\,G_1)$ is controllable and the matrix F_2 is Hurwitz. It follows that there exist matrices C_1, P_1 and P_2 and strictly positive constants α_1 and α_2 such that:

$$\begin{aligned} P_1\,(F_1 - G_1 C_1) \;+\; (F_1 - G_1 C_1)^{\mathrm{T}}\, P_1 \;&=\; -I \;\leq\; -2\alpha_1\, P_1 \\ P_1\, F_2 \;+\; F_1^{\mathrm{T}} P_2 \;&=\; -I \;\leq\; -2\alpha_2\, P_2\,. \end{aligned} \tag{310}$$

Let us now define a function U by:

$$U(\chi) \;=\; \frac{\left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)^j}{2j} \;+\; \frac{\left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)}{2} \;+\; \beta\, \frac{\left(\chi_2^{\mathrm{T}} P_2 \chi_2\right)}{2}\;, \tag{311}$$

where β is a strictly positive constant. By letting:

$$V(x,p) \;=\; U\left(\Phi(p)\, x\right), \tag{312}$$

assumptions BO and CO are clearly satisfied. To check that assumption URS holds also, we choose u_{n} as:

$$u_{\mathrm{n}}(x,p) \;=\; -C_1\, \chi_1 \;-\; L(y)\,, \qquad \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \;=\; \Phi(p)\, x\,. \tag{313}$$

Then along the solutions of:

$$\dot{\chi} \;=\; \begin{pmatrix} F_1 - G_1 C_1 & 0 \\ 0 & F_2 \end{pmatrix} \chi \;+\; \begin{pmatrix} 0 \\ G_2 \end{pmatrix} L\left(H_1 \chi_1 + \delta(t)\right), \tag{314}$$

the time derivative of U satisfies:

$$\begin{aligned}\dot{U} = &\left[\left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)^{j-1} + 1\right] \chi_1^{\mathrm{T}} P_1 \left(F_1 - G_1 C_1\right) \chi_1 \\ &+ \beta \left[\chi_2^{\mathrm{T}} P_2 F_2 \chi_2 + \chi_2^{\mathrm{T}} P_2 G_2 L \left(H_1 \chi_1 + \delta(t)\right)\right] \\ \leq &-\alpha_1 \left[\left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)^j + \left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)\right] - \alpha_2 \beta \left(\chi_2^{\mathrm{T}} P_2 \chi_2\right) \\ &+ \beta \left(\chi_2^{\mathrm{T}} P_2 \chi_2\right)^{\frac{1}{2}} |L| \sqrt{G_2{}^{\mathrm{T}} P_2 G_2}\,. \end{aligned} \tag{315}$$

But the growth condition (305), satisfied by the function L, and Young's inequality imply the existence of four positive constants γ_1 to γ_4 such that, for all χ_1, χ_2 and δ, we have:

$$\begin{aligned}&\left(\chi_2^{\mathrm{T}} P_2 \chi_2\right)^{\frac{1}{2}} |L(H_1 \chi_1 + \delta)| \sqrt{G_2{}^{\mathrm{T}} P_2 G_2} \\ &\leq \frac{\alpha_2}{2} \left(\chi_2^{\mathrm{T}} P_2 \chi_2\right) + \gamma_1 \left(\chi_1^{\mathrm{T}} P_1 \chi_1\right) + \gamma_2 |\delta|^2 + \gamma_3 \left(\chi_1^{\mathrm{T}} P_1 \chi_1\right)^j + \gamma_4 |\delta|^{2j}\,. \end{aligned} \tag{316}$$

We have established:

$$\dot{U} \leq -c\, U + \beta \left[\gamma_2 |\delta|^2 + \gamma_4 |\delta|^{2j}\right], \tag{317}$$

where:

$$c = \min\left\{2(\alpha_1 - \beta\gamma_1)\,,\, 2j(\alpha_1 - \beta\gamma_3)\,,\, \beta\alpha_2\right\}\,. \tag{318}$$

It follows that assumption URS holds, up to the presence of the exponentially decaying terms in δ, if β is chosen sufficiently small. □

Example: Introduction to System (320) (319)

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + p^* x_1^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\,. \end{aligned} \tag{320}$$

Following the Lyapunov design proposed in [27], we choose:

$$\begin{aligned}u_{\mathrm{n}}(p,x) = &-\left(\frac{1}{2k} + \frac{1}{2} + 2p\chi_1\right)\left(\chi_3 - \frac{\chi_2}{2} - \chi_1^{2k-1}\right) \\ &+ \left[\frac{1}{4k^2} + \frac{2p\chi_1}{k} - (2k-1)\chi_1^{2k-2} - 2p\chi_2\right]\left(\chi_2 - \frac{\chi_1}{2k}\right) \\ &- \frac{\chi_3}{2} - \chi_2 \left(\frac{\chi_2^2}{2} + \frac{\chi_1^{2k}}{2k}\right)^{j-1}, \end{aligned} \tag{321}$$

and:

$$V(p,x) = U(\chi) = \frac{\chi_3^2}{2} + \frac{1}{j}\left(\frac{\chi_2^2}{2} + \frac{\chi_1^{2k}}{2k}\right)^j, \tag{322}$$

where k and j are integers larger or equal to 1 and χ is given by:

$$\chi = \begin{pmatrix}\chi_1 \\ \chi_2 \\ \chi_3\end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + \dfrac{x_1}{2k} + p\, x_1^2 \\ x_3 + \dfrac{1}{2}\left(x_2 + \dfrac{x_1}{2k} + p x_1^2\right) \\ \quad + \left(\dfrac{1}{2k} + 2p x_1\right)\left(x_2 + p x_1^2\right) + x_1^{2k-1} \end{pmatrix}. \tag{323}$$

Note that, if $k = j = 1$, u_{n} is a linearizing feedback.

Since V is positive definite and radially unbounded, assumptions BO and CO are satisfied. Assumption URS holds also, since a straightforward computation shows that the time derivative of V along the solutions of (320) with p instead of p^* satisfies:

$$\dot{V} \;=\; -V\,. \tag{324}$$

□

Proposition **(325)**

Let assumptions BO, URS and ICS hold and choose:

$$h(x,p) \;=\; \frac{\alpha_1\, V(x,p)}{\alpha_1 - V(x,p)} \qquad \text{and} \qquad r(e,x,p) \;=\; 1 + \left(\frac{\alpha_1\, V(x,p)}{\alpha_1 - V(x,p)}\right)^2 . \tag{326}$$

If assumption $\varLambda$-LP is satisfied with:

$$\varLambda(x,t) \;=\; \frac{\partial h}{\partial x}(x,\widehat{p}(t))\,, \tag{327}$$

and:

either *assumption MC holds, α_1 is chosen smaller than or equal to α_0, $x(0)$ belongs to $\varOmega_0$ and $V(x(0),\widehat{p}(0)) < \alpha_1$,*

or *we are in the global case, i.e., $\varOmega_0 = \varOmega = \mathbb{R}^n$, $\alpha_1 = \alpha_0 = +\infty$ and there exists a C^0 function $d_1 : \varPi \to \mathbb{R}_+$ such that, for all (x,p) in $\mathbb{R}^n \times \varPi$:*

$$\left\|\frac{\partial V}{\partial p}(x,p)\right\| \cdot \left\|\left[\frac{\partial V}{\partial x}(x,p)\, A(x,u_{\mathrm{n}}(x,p))\right]^{\mathrm{T}}\right\| \;\leq\; d_1(p)\, \max\{1\,,\, V(x,p)^2\}\,, \tag{328}$$

then all the corresponding solutions $(x(t),\widehat{p}(t),\widehat{h}(t))$ of (S_{p^*})-(230)-(231)-(232) are well-defined on $[0,+\infty)$, unique, bounded and:

$$\lim_{t\to\infty} V(x(t),\widehat{p}(t)) \;=\; 0\,. \tag{329}$$

It follows that the Adaptive Stabilization problem is solved if assumption CO holds also.

Proposition (325) is an extension to the case when V is not radially unbounded of a result established by Pomet and Praly [23, 21].

Proof.
Case: Assumption MC holds: The system we consider is, with notation (13):

$$\begin{aligned}
\dot{x} &= a\left(x\,,\, u_{\mathrm{n}}(x,\widehat{p}) \;+\; v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right) + A\left(x\,,\, u_{\mathrm{n}}(x,\widehat{p}) \;+\; v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right)\, p^* \\
\dot{\widehat{h}} &= -r(e,x,\widehat{p})\, e \;+\; \frac{\partial h}{\partial x}(x,\widehat{p})\, [a_0(x) + A_0(x)\widehat{p} + u_{\mathrm{n}}(x,\widehat{p}) \odot (b(x) + B(x)\widehat{p})] \\
&\qquad + \frac{\partial h}{\partial x}(x,\widehat{p})\, v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot (b(x) + B(x)\widehat{q}) + \frac{\partial h}{\partial p}(x,\widehat{p})\, \dot{\widehat{p}} \\
e &= \widehat{h} \;-\; h(x,\widehat{p}) \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(I, \widehat{p}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\, (A_0(x) + u_{\mathrm{n}}(x,\widehat{p}) \odot B(x))\right]^{\mathrm{T}} e\right)\,,\; \widehat{p}(0) \in \varPi_1 \\
\dot{\widehat{q}} &= \mathrm{Proj}\left(I, \widehat{q}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\, v(x,\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot B(x)\right]^{\mathrm{T}} e\right)\,,\; \widehat{q}(0) \in \varPi_1\,,
\end{aligned} \tag{330}$$

with:

$$h(x,p) \;=\; \frac{\alpha_1\, V(x,p)}{\alpha_1 - V(x,p)}\;, \qquad r(e,x,p) \;=\; 1 \,+\, h(x,p)^2\,, \tag{331}$$

and, with notation (13):

$$\frac{\partial V}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} + \frac{\partial V}{\partial x}(x,\widehat{p})\,v(x,\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot [b(x) + B(x)\,\widehat{q}] \;=\; 0\,. \tag{332}$$

¿From our smoothness assumptions on the various functions and with Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{p},\widehat{q},\widehat{h}) \in \Omega \times \Pi \times \Pi \times \mathbb{R} \qquad \text{and} \qquad V(x,\widehat{p}) \;<\; \alpha_1\,. \tag{333}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{p}(t),\widehat{q}(t),\widehat{h}(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite, and satisfying (333) for all t in $[0,T)$ and in particular:

$$V(x(t),\widehat{p}(t)) \;<\; \alpha_1 \quad \forall\, t \,\in\, [0,T)\,. \tag{334}$$

Applying Points 3 and 4 of Lemma (235), we also know that, for all t in $[0,T)$:

$$\begin{aligned}
&\widehat{p}(t) \,\in\, \Pi_1 \quad \text{and} \quad \widehat{q}(t) \,\in\, \Pi_1 \\
&\left\| \begin{matrix} \widehat{p}(t) - p^* \\ \widehat{q}(t) - p^* \end{matrix} \right\|^2 + \|e(t)\|^2 \,+\, 2\int_0^t r\,\|e\|^2 \;\leq\; \left\| \begin{matrix} \widehat{p}(0) - p^* \\ \widehat{q}(0) - p^* \end{matrix} \right\|^2 + \|e(0)\|^2 \;\stackrel{\text{def}}{=}\; \beta^2\,.
\end{aligned} \tag{335}$$

Then, with assumption URS, (332), (334) and (331), we get successively from the $\dot{\widehat{h}}$ equation in (330):

$$\begin{aligned}
\dot{\widehat{h}} &\leq -r\,e \,-\, c\,\frac{{\alpha_1}^2\, V}{(\alpha_1 - V)^2} && (336)\\
&\leq \sqrt{r}\,\left(\sqrt{r}\,|e|\right) \,-\, c\,\frac{\alpha_1}{\alpha_1 - V}\,h && (337)\\
&\leq (1+h)\,\left(\sqrt{r}\,|e|\right) \,-\, c\,h && (338)\\
&\leq -\left(c - \left(\sqrt{r}\,|e|\right)\right) h \,+\, \sqrt{r}\,|e| && (339)\\
&\leq -\left(c - \left(\sqrt{r}\,|e|\right)\right) \widehat{h} \,+\, (1+|e|)\,\sqrt{r}\,|e| \,+\, c\,|e|\,. && (340)
\end{aligned}$$

Now, inequality (335) implies that the assumption of Lemma (583) in Appendix B is satisfied with:

$$X \;=\; \widehat{h}\,, \tag{341}$$

and, by using the fact that $r \geq 1$:

$$\begin{aligned}
&\vartheta_1 \;=\; \sqrt{r}\,|e|\,, && \sigma_1 \;=\; 2,\; S_{11} \;=\; \frac{\beta^2}{2} \\
&\varpi_1 \;=\; (1+\beta)\,\sqrt{r}\,|e| \,+\, c\,|e|\,,\; \zeta_1 \;=\; 2,\; && S_{21} \;=\; 2\left((1+\beta)^2 + c^2\right)\frac{\beta^2}{2}\;.
\end{aligned} \tag{342}$$

It follows that there exists a constant Υ, depending only on the initial condition, such that, for all t in $[0,T)$, we have:

$$0 \;\leq\; \frac{\alpha_1\, V(x(t),\widehat{p}(t))}{\alpha_1 - V(x(t),\widehat{p}(t))} \;=\; h(x(t),\widehat{p}(t)) \;=\; \widehat{h}(t) \,-\, e(t) \;\leq\; \Upsilon \,+\, \beta\,. \tag{343}$$

Hence, we have established, for all t in $[0,T)$:

$$\begin{aligned} V(x(t),\widehat{p}(t)) \;&\leq\; \frac{\alpha_1(\varUpsilon+\beta)}{\alpha_1+\varUpsilon+\beta} \;<\; \alpha_1 \\ -\beta \;\leq\; \widehat{h}(t) \;&\leq\; \varUpsilon \\ \|\widehat{p}(t)-p^*\| \;\leq\; \beta \qquad &\text{and} \qquad \widehat{p}(t)\;\in\;\varPi_1 \\ \|\widehat{q}(t)-p^*\| \;\leq\; \beta \qquad &\text{and} \qquad \widehat{q}(t)\;\in\;\varPi_1\,. \end{aligned} \tag{344}$$

Then, from assumption BO, we know the existence of a compact subset $\varGamma$ of $\varOmega$ such that:

$$x(t)\;\in\;\varGamma \quad \forall\, t\;\in\;[0,T)\,. \tag{345}$$

Hence, the solution remains in a compact subset of the open set defined in (333). It follows by contradiction that $T=+\infty$ and that $x(t)$ and $\widehat{p}(t)$, $u(t)$ and $\dot{\widehat{p}}(t)$ are bounded on $[0,+\infty)$. Then, using the second conclusion of Lemma (583), we have:

$$\limsup_{t\to+\infty}\,\widehat{h}(t)\;\leq\;0\,. \tag{346}$$

Also, from (229) and the fact that the solution is bounded, we deduce that $\dot{e}$ is bounded. Since, from (335), e is in $L^2([0,+\infty))$, we have established:

$$\lim_{t\to+\infty} e(t)\;=\;0\,. \tag{347}$$

This yields:

$$0\;\leq\;\limsup_{t\to+\infty}\,h(x(t),\widehat{p}(t))\;=\;\limsup_{t\to+\infty}\,\widehat{h}(t)\;-\;\lim_{t\to+\infty} e(t)\;\leq\;0\,. \tag{348}$$

With the definition of h, (334), and assumption CO, this implies:

$$\lim_{t\to+\infty} x(t)\;=\;\mathcal{E}\,. \tag{349}$$

Case: Inequality (328) holds: The system we consider is:

$$\begin{aligned} \dot{x} &= a\,(x\,,\,u_{\rm n}(x,\widehat{p}))+A\,(x\,,\,u_{\rm n}(x,\widehat{p}))\;p^* \\ \dot{\widehat{h}} &= -r(e,x,\widehat{p})\,e\;+\;\frac{\partial V}{\partial x}(x,\widehat{p})\,A(x,u_{\rm n}(x,\widehat{p}))\,\widehat{p}\;+\;\frac{\partial V}{\partial x}(x,\widehat{p})\,a(x,u_{\rm n}(x,\widehat{p})) \\ &\qquad +\;\frac{\partial V}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \\ e &= \widehat{h}\;-\;V(x,\widehat{p}) \\ \dot{\widehat{p}} &= \mathrm{Proj}\left(I\,,\,\widehat{p}\,,\,-\left(\frac{\partial V}{\partial x}(x,\widehat{p})\,A(x,u_{\rm n}(x,\widehat{p}))\right)^{\rm T} e\right), \qquad \widehat{p}(0)\;\in\;\varPi_1\,, \end{aligned} \tag{350}$$

with:

$$r(e,x,\widehat{p})\;=\;1\;+\;V(x,\widehat{p})^2\,. \tag{351}$$

¿From our smoothness assumptions on the various functions and with Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side for all $(x,\widehat{p},\widehat{h})$ in $\mathbb{R}^n\times\varPi\times\mathbb{R}$. It follows that, for any initial condition in this open set, there exists a unique solution $(x(t), \widehat{p}(t), \widehat{h}(t))$, defined on a right maximal interval $[0, T)$, with T possibly infinite, and remaining in this set. Applying Points 1 and 2 of Lemma (235), we also know that, for all t in $[0, T)$:

$$\begin{aligned}&\widehat{p}(t) \in \Pi_1\\&\|\widehat{p}(t) - p^*\|^2 + \|e(t)\|^2 + 2\int_0^t r\,\|e\|^2 \le \|\widehat{p}(0) - p^*\|^2 + \|e(0)\|^2 \stackrel{\text{def}}{=} \beta^2 .\end{aligned} \tag{352}$$

Then, as in the previous case, with assumption URS, we get from the $\dot{\widehat{h}}$ equation in (350):

$$\dot{\widehat{h}} \le -r\,e - c\,V + \frac{\partial V}{\partial p}\,\dot{\widehat{p}} . \tag{353}$$

But, by using inequality (328), Point 2 of Lemma (103), (351) and the expression of $\dot{\widehat{p}}$ in (350), we get successively:

$$\begin{aligned}\left\|\frac{\partial V}{\partial p}\,\dot{\widehat{p}}\right\| &\le \left\|\frac{\partial V}{\partial p}\right\| \left\|\operatorname{Proj}\left(I\,,\,\widehat{p}\,,\,-\left(\frac{\partial V}{\partial x}\,A\right)^{\mathrm{T}} e\right)\right\|\\&\le \left\|\frac{\partial V}{\partial p}\right\| \left\|\left[\frac{\partial V}{\partial x}\,A\right]^{\mathrm{T}}\right\| |e|\\&\le d_1(\widehat{p}(t))\,\max\{1\,,\,V^2\}\,|e|\\&\le d_1(\widehat{p}(t))\left(1+V^2\right)\,|e|\\&\le d_1(\widehat{p}(t))\,r\,|e| .\end{aligned} \tag{354}$$

Since d_1 depends continuously on $\widehat{p}(t)$ which satisfies (352), there exists a constant k depending only on $\widehat{p}(0)$ and $e(0)$ such that (353), (354) and the expression of e in (350) give:

$$\begin{aligned}\dot{\widehat{h}} &\le (1+k)\,\sqrt{r}\,\left(\sqrt{r}\,|e|\right) - c\,V\\&\le (1+k)\,(1+\widehat{h}-e)\,\left(\sqrt{r}\,|e|\right) - c\,(\widehat{h}-e)\\&\le -\left(c-(1+k)\sqrt{r}|e|\right)\,\widehat{h} + (1+k)\,(1+|e|)\,\left(\sqrt{r}\,|e|\right) + c\,|e| .\end{aligned} \tag{355}$$

Hence, with Point 2 of Lemma (235), the assumption of Lemma (583) in Appendix B is satisfied with (see (352)):

$$X = \widehat{h} \tag{356}$$

and

$$\begin{aligned}&\vartheta_1 = (1+k)\sqrt{r}|e|\,, && \sigma_1 = 2,\ S_{11} = (1+k)^2\,\frac{\beta^2}{2}\\&\varpi_1 = (1+k)(1+\beta)\left(\sqrt{r}|e|\right) + c|e|, && \zeta_1 = 2,\ S_{21} = 2\left((1+k)^2(1+\beta)^2 + c^2\right)\frac{\beta^2}{2} .\end{aligned} \tag{357}$$

¿From here, we conclude the proof exactly as in the previous case. □

Compared with the Lyapunov design, we see that, when the Matching Condition (MC) holds, the equation error filtering technique, with V as the observation function, requires the more restrictive Uniform Reduced-order Stabilizability (URS) assumption. However, if assumption MC does not hold, nothing is known for the Lyapunov design, whereas here the Adaptive Stabilization problem is solved in the global case by the equation error filtering technique if the quadratic growth condition (328) is satisfied.

Example: System (320) Continued (358)
For the system (320), we have shown that assumption BO, CO and URS are satisfied. For assumption MC, we get (see (322)):

$$\frac{\partial V}{\partial p} = \chi_2 \chi_1^2 \left(\frac{\chi_2^2}{2} + \frac{\chi_1^{2k}}{2k}\right)^{j-1} + \chi_3 \left[\frac{\chi_1^2}{2} + 2\chi_1 \left(\chi_2 - \frac{\chi_1}{2k}\right) + \left(\frac{1}{2k} + 2p\chi_1\right) \chi_1^2\right], \quad (359)$$

and:

$$\frac{\partial V}{\partial x} b = \chi_3 \,. \quad (360)$$

Hence, (170) cannot be satisfied, since, when $\frac{\partial V}{\partial x} b$ is zero, $\frac{\partial V}{\partial p}$ is not necessarily zero. Then, let us see if the growth condition (328) holds. We have to compare the product of the norms of (see (322)):

$$\frac{\partial V}{\partial x} A = \frac{\partial U}{\partial \chi} \frac{\partial \chi}{\partial x_1} x_1^2 \qquad \text{and} \qquad \frac{\partial V}{\partial p} = \frac{\partial U}{\partial \chi} \frac{\partial \chi}{\partial p}$$

to a power of $V = U$. We have, writing everything in the χ coordinates in which U has a simpler expression:

$$\frac{\partial U}{\partial \chi} = \left(\left(\frac{\chi_2^2}{2} + \frac{\chi_1^{2k}}{2k}\right)^{j-1} \chi_1^{2k-1}, \quad \left(\frac{\chi_2^2}{2} + \frac{\chi_1^{2k}}{2k}\right)^{j-1} \chi_2 , \quad \chi_3 \right) \quad (361)$$

$$\frac{\partial \chi}{\partial x_1} = \begin{pmatrix} 1 \\ \frac{1}{2k} + 2p\chi_1 \\ \left(\frac{1}{2k} + 2p\chi_1\right)\left(\frac{1}{2} + 2p\chi_1\right) + 2p\left(\chi_2 - \frac{\chi_1}{2k}\right) + (2k-1)\,\chi_1^{2k-2} \end{pmatrix} \quad (362)$$

$$\frac{\partial \chi}{\partial p} = \begin{pmatrix} 0 \\ \chi_1^2 \\ \frac{\chi_1^2}{2} + 2\chi_1\left(\chi_2 - \frac{\chi_1}{2k}\right) + \left(\frac{1}{2k} + 2p\chi_1\right)\chi_1^2 \end{pmatrix}. \quad (363)$$

To obtain our inequalities, we note that:

$$|\chi_1| \leq (\gamma U)^{\frac{1}{2kj}} , \quad |\chi_2| \leq (\gamma U)^{\frac{1}{2j}} , \quad |\chi_3| \leq (\gamma U)^{\frac{1}{2}} , \quad (364)$$

with:

$$\gamma = \sup \left\{ j(2k)^j , \, j2^j , \, 2 \right\} , \quad (365)$$

and, for any positive α:

$$|a + b\,U^\alpha| \leq (|a| + |b|) \sup\{1 , U^\alpha\} \,. \quad (366)$$

We get:

$$\left\|\frac{\partial V}{\partial x}A\right\| \leq d_1(p)\,\sup\{1\,,\,U^{\alpha_1}\} \quad \text{and} \quad \left\|\frac{\partial V}{\partial p}\right\| \leq d_2(p)\,\sup\{1\,,\,U^{\alpha_2}\}\,, \qquad (367)$$

with:

$$\begin{aligned} d_1(p) &= \left[(c_1+2\,|p|)+(c_1c_2+2c_2\,|p|+4p^2)+2\,|p|+(2k-1)\right]\gamma^{\alpha_1} \\ \alpha_1 &= \sup\left\{1+\frac{1}{2kj}\,,\,1-\frac{1}{2j}+\frac{3}{2kj}\,,\,\frac{1}{2}+\frac{2}{kj}\,,\,\frac{1}{2}+\frac{1}{2j}+\frac{1}{kj}\,,\,\frac{1}{2}+\frac{k}{kj}\right\} \\ d_2(p) &= \left[2+|c_1-c_2|+2\,|p|\right]\gamma^{\alpha_2} \qquad (368) \\ \alpha_2 &= \sup\left\{1-\frac{1}{2j}+\frac{1}{kj}\,,\,\frac{1}{2}+\frac{3}{2kj}\,,\,\frac{1}{2}+\frac{k+1}{2kj}\right\}. \qquad (369) \end{aligned}$$

It follows that:

$$\left\|\frac{\partial V}{\partial x}(p,x)\,A\,(x,u_{\mathrm{n}}(p,x))\right\|\,\left\|\frac{\partial V}{\partial p}(p,x)\right\| \leq d_1(p)\,d_2(p)\,\left(1+V(p,x)^{\alpha}\right), \qquad (370)$$

where α depending on j and k is given in Table 1. Hence, for this example, (328) is satisfied if we choose $k > 2$ and $j > 1$. □

Table 1. $\alpha(k,j)$

k \ j	1	2	3	4	5
1	9/2	11/4	5/2	19/8	23/10
2	11/4	17/8	25/12	33/16	41/20
3	8/3	2	2	2	2
4	21/8	31/16	47/24	63/32	79/40
5	13/5	19/10	29/15	39/20	49/25

4.2 Regressor Filtering

When the regressor filtering (258)–(261) is used, using (259) and (261) we get:

$$\begin{aligned} \dot{\widehat{h}} &= -\rho(x,\widehat{p},u)\,e + Z_{\mathrm{f}}\,\dot{\widehat{p}} \\ &\quad + \frac{{\alpha_1}^2}{(\alpha_1-V(x,\widehat{p}))^2}\left(\frac{\partial V}{\partial x}(x,\widehat{p})\,(a(x,u)+A(x,u)\widehat{p}) + \frac{\partial V}{\partial p}(x,\widehat{p})\dot{\widehat{p}}\right), \qquad (371) \end{aligned}$$

where:

$$\widehat{h} = h(x,\widehat{p}) + e = Z_{\mathrm{f}}\,\widehat{p} + \widehat{z}_{\mathrm{f}}\,. \qquad (372)$$

Hence, if:

$$u = u_{\mathrm{n}}(x,\widehat{p})\,, \qquad (373)$$

and (47) in assumption PRS holds, we have:

$$\dot{\widehat{h}} \leq -\rho(x,\widehat{p},u)\,e + \left(\frac{{\alpha_1}^2}{(\alpha_1 - V(x,\widehat{p}))^2}\,\frac{\partial V}{\partial p}(x,\widehat{p}) + Z_{\mathrm{f}}\right)\dot{\widehat{p}}\,. \tag{374}$$

Compared with equation (293) for the equation error filtering case, we have the extra term $Z_{\mathrm{f}}\,\dot{\widehat{p}}$. But, thanks to Lemma (264), we know that we may choose ρ and $\dot{M}$ in order to guarantee that Z_{f} is bounded and $\dot{\widehat{p}}$ is absolutely integrable. It follows that the following counterpart to Proposition (325) can be established:

Proposition **(375)**
Let assumptions BO, URS and ICS hold. Choose:

$$\begin{aligned} h(x,p) &= \frac{\alpha_1\,V(x,p)}{\alpha_1 - V(x,p)} \\ \rho(x,p,u) &= 1 + \left\|\frac{{\alpha_1}^2}{(\alpha_1 - V(x,p))^2}\,\frac{\partial V}{\partial x}(x,p)\,A(x,u)\right\| \\ r(x,p,u,e) &= \rho(x,p,u)^2\,, \end{aligned} \tag{376}$$

and:

$$\dot{M} \;=\; G\,(M\,,\,Z_{\mathrm{f}}\,,\,r)\,, \qquad \varepsilon_3\,M(0)\;>\;I\,, \tag{377}$$

where G is a negative symmetric matrix, depending locally-Lipschitz-continuously on its arguments and satisfying (see (262) and (263)):

$$-\varepsilon_4\,M\,Z_{\mathrm{f}}^{\mathrm{T}}\,Z_{\mathrm{f}}\,M\,r \;\geq\; G \;\geq\; -(2-\varepsilon_1)\,M\,Z_{\mathrm{f}}^{\mathrm{T}}\,Z_{\mathrm{f}}\,M\,r\,. \tag{378}$$

If assumption Λ-LP is satisfied with:

$$\Lambda(x,t) \;=\; \frac{\partial h}{\partial x}(x,\widehat{p}(t))\,, \tag{379}$$

and:

either *assumption MC holds and, with notation (13), $B(x)\equiv 0$, α_1 is chosen smaller than or equal to α_0, $x(0)$ belongs to Ω_0 and $V(x(0),\widehat{p}(0)) < \alpha_1$,*

or *we are in the global case, i.e., $\Omega_0 = \Omega = \mathbb{R}^n$, $\alpha_1 = \alpha_0 = +\infty$ and there exists a C^0 function $d_2 : \Pi \to \mathbb{R}_+$ such that, for all (x,p) in $\mathbb{R}^n \times \Pi$:*

$$\left\|\frac{\partial V}{\partial p}(x,p)\right\| \;\leq\; d_2(p)\,\max\{1\,,\,V(x,p)\}\,, \tag{380}$$

then all the corresponding solutions $(x(t),\widehat{p}(t),\widehat{z}_{\mathrm{f}}(t),Z_{\mathrm{f}}(t)),M(t)$ of (S_{p^*}), (259), (261), (262) and (377) are well-defined on $[0,+\infty)$, unique, bounded and:

$$\lim_{t\to\infty} V(x(t),\widehat{p}(t)) \;=\; 0\,. \tag{381}$$

It follows that the Adaptive Stabilization problem is solved if assumption CO holds also.

Proof.
Case: Assumption MC holds: In this case, we assume also that $A(x,u)$ does not depend on u, i.e., with notation (13), we have:

$$A(x,u) \;=\; A_0(x)\,. \tag{382}$$

The system we consider is:

$$\begin{aligned}
\dot{x} &= a\,(x\,,\,u) + A_0(x)\,p^* \\
\dot{\widehat{z_{\mathrm{f}}}} &= \rho(x,\widehat{p})\,z_{\mathrm{f}} \,+\, \frac{\partial h}{\partial x}(x,\widehat{p})\,a(x,u) \,+\, \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}}\,, \quad \widehat{z_{\mathrm{f}}}(0) \;=\; h(x(0),\widehat{p}(0)) \\
z_{\mathrm{f}} &= h(x,\widehat{p}) \,-\, \widehat{z_{\mathrm{f}}} \\
\dot{Z}_{\mathrm{f}} &= -\rho(x,\widehat{p})\,Z_{\mathrm{f}} \,+\, \frac{\partial h}{\partial x}(x,\widehat{p})\,A_0(x)\,, \quad Z_{\mathrm{f}}(0) \;=\; 0 \\
e &= Z_{\mathrm{f}}\,\widehat{p} \,-\, z_{\mathrm{f}} \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(M\,,\,\widehat{p}\,,\,-M\,Z_{\mathrm{f}}^{\mathrm{T}}\,r(x,\widehat{p})\,e\right), \quad \widehat{p}(0) \;\in\; \mathit{\Pi}_1 \\
\dot{M} &= G\,(M\,,\,Z_{\mathrm{f}}\,,\,r(x,\widehat{p}))\;, \quad M(0) \;>\; 0\,,
\end{aligned} \tag{383}$$

with:

$$u = u_{\mathrm{n}}(x,\widehat{p}) + v(x,\widehat{p},\dot{\widehat{p}})\;, \quad h(x,p) = \frac{\alpha_1\,V(x,p)}{\alpha_1 - V(x,p)}\;, \quad r(x,p) = \rho(x,p)^2\,, \tag{384}$$

and

$$\rho(x,p) = 1 + \left\|\frac{{\alpha_1}^2}{(\alpha_1 - V(x,p))^2}\,\frac{\partial V}{\partial x}(x,p)\,A_0(x))\right\|\,, \tag{385}$$

where:

$$\frac{\partial V}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} + \frac{\partial V}{\partial x}(x,\widehat{p})\,v(x,\widehat{p},\dot{\widehat{p}}) \odot b(x) \;=\; 0\,. \tag{386}$$

Since r does not depend on u, $\dot{\widehat{p}}$ is explicitly defined. Then, from our smoothness assumptions on the various functions and with Lemma (103), this system (383) has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{z_{\mathrm{f}}},Z_{\mathrm{f}},\widehat{p},M) \;\in\; \mathit{\Omega}\times\mathbb{R}\times\mathbb{R}^l\times\mathit{\Pi}\times\mathcal{M} \qquad \text{and} \qquad V(x,\widehat{p}) \;<\; \alpha_1\,. \tag{387}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{z_{\mathrm{f}}}(t),Z_{\mathrm{f}}(t),\widehat{p}(t),M(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite, and satisfying (387). Applying Points 1 and 2 of Lemma (264), we also know that, for all t in $[0,T)$:

$$\begin{aligned}
&\widehat{p}(t) \;\in\; \mathit{\Pi}_1 \\
&\varepsilon_2\,\|\widehat{p}(t) - p^*\|^2 \,+\, \varepsilon_1 \int_0^t r\;\|e\|^2 \;\leq\; \varepsilon_3\,\|\widehat{p}(0) - p^*\|^2 \;\stackrel{\mathrm{def}}{=}\; \beta^2\,.
\end{aligned} \tag{388}$$

Moreover, our choice for ρ implies with the last statement of Lemma (264) that, for all t in $[0,T)$:

$$\|Z_{\mathrm{f}}\| \;\leq\; 1 \qquad \text{and} \qquad \|e\| \;\leq\; \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\,\|\widehat{p}(0) - p^*\|\;. \tag{389}$$

Then, since we have (see (260)):

$$z_{\mathrm{f}} = Z_{\mathrm{f}}\, p^* \,, \tag{390}$$

we conclude also that for all t in $[0,T)$:

$$\|z_{\mathrm{f}}\| \leq \|p^*\| \,. \tag{391}$$

Finally, since $M(t) \in \mathcal{M}$ for all t in $[0,T)$, (377) and (378) imply that $\dot{M}$ is negative and:

$$\dot{\overline{M^{-1}}} \leq (2-\varepsilon_1)\, Z_{\mathrm{f}}^{\mathrm{T}} Z_{\mathrm{f}}\, r \,. \tag{392}$$

With (389) and (377), this implies, for all t in $[0,T)$:

$$\frac{I}{\varepsilon_3 + (2-\varepsilon_1) \int_0^T r} \leq M(t) \leq M(0) \,. \tag{393}$$

Then, for $\widehat{h}$ defined in (372), we get by (371) and (298) in assumption URS:

$$\dot{\widehat{h}} \leq -\rho\, e - c\, \frac{{\alpha_1}^2\, V}{(\alpha_1 - V)^2} + Z_{\mathrm{f}}\, \dot{\widehat{p}} \tag{394}$$

$$\leq -\left(\sqrt{r}\, |e|\right) - c\, h + Z_{\mathrm{f}}\, \dot{\widehat{p}} \tag{395}$$

$$\leq -c\, \widehat{h} + \sqrt{r}\, |e| + c\, |e| + Z_{\mathrm{f}}\, \dot{\widehat{p}} . \tag{396}$$

With (388) and Point 3 of Lemma (264), the assumption of Lemma (583) in Appendix B is satisfied with:

$$X = \widehat{h} \tag{397}$$

and:

$$\begin{aligned} \varpi_1 &= \sqrt{r}\, |e| + c\, |e| \,, & \zeta_1 &= 2,\ S_{21} = 2\left(1+c^2\right) \frac{\beta^2}{\varepsilon_1} \\ \varpi_2 &= \left\| \dot{\widehat{p}} \right\| , & \zeta_2 &= 1,\ S_{22} = k_1(p^*, \widehat{p}(0)) \,. \end{aligned} \tag{398}$$

It follows that there exists a constant Υ depending only on the initial conditions such that, for all t in $[0,T)$, we have:

$$0 \leq \frac{\alpha_1\, V(x(t), \widehat{p}(t))}{\alpha_1 - V(x(t), \widehat{p}(t))} = h(x(t), \widehat{p}(t)) \tag{399}$$

$$= \widehat{h}(t) - e(t) \tag{400}$$

$$\leq \Upsilon + \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\, \|\widehat{p}(0) - p^*\| \,, \tag{401}$$

where we have used (389). Hence, we have established, for all t in $[0,T)$:

$$\begin{gathered} V(x(t), \widehat{p}(t)) \leq \frac{\alpha_1\, (\Upsilon + \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\, \|\widehat{p}(0) - p^*\|)}{\alpha_1 + \Upsilon + \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\, \|\widehat{p}(0) - p^*\|} < \alpha_1 \\ \|\widehat{p}(t) - p^*\| \leq \frac{\beta}{\sqrt{\varepsilon_2}} \qquad \text{and} \qquad \widehat{p}(t) \in \Pi_1 \\ \|Z_{\mathrm{f}}(t)\| \leq 1 \\ \|z_{\mathrm{f}}(t)\| \leq \|p^*\| \,. \end{gathered} \tag{402}$$

Then, from assumption BO, we know the existence of a compact subset Γ of Ω such that:

$$x(t) \in \Gamma \quad \forall t \in [0,T). \tag{403}$$

We have also:

$$\frac{I}{\varepsilon_3 + (2-\varepsilon_1)\,T\,R} \le M(t) \le M(0), \tag{404}$$

where R is an upper bound for $\int_0^T r$ whose existence follows from the continuity of the function r and the boundedness of x and $\widehat{p}$. Finally, with the continuity of the function h and the fact that:

$$\widehat{z_{\mathrm{f}}} = h(x,\widehat{p}) - z_{\mathrm{f}}, \tag{405}$$

we have proved that the solution remains in a compact subset of the open set defined in (387). It follows by contradiction that $T = +\infty$ and in particular that $x(t)$, $\widehat{p}(t)$, $u(t)$ and $\dot{\widehat{p}}(t)$ are bounded on $[0,+\infty)$. ¿From here, we conclude the proof exactly as in the proof of Proposition (325).

Case: Inequality (380) holds: The system we consider is the system (383) with:

$$u = u_{\mathrm{n}}(x,\widehat{p}), \quad h(x,p) = V(x,p), \tag{406}$$

$$\rho(x,p,u) = 1 + \left\|\frac{\partial V}{\partial x}(x,p)\,A(x,u))\right\| \quad \text{and} \quad r(x,p,u,e) = \rho(x,p,u)^2. \tag{407}$$

¿From our smoothness assumptions on the various functions and with Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{z_{\mathrm{f}}},Z_{\mathrm{f}},\widehat{p},M) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \times \Pi \times \mathcal{M}. \tag{408}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{z_{\mathrm{f}}}(t),Z_{\mathrm{f}}(t),\widehat{p}(t),M(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite and satisfying (408). Then, as in the previous case, we have, for all t in $[0,T)$:

$$\begin{aligned}
&\widehat{p}(t) \in \Pi_1 \\
&\varepsilon_2\,\|\widehat{p}(t)-p^*\|^2 + \varepsilon_1 \int_0^t r\,\|e\|^2 \le \varepsilon_3\,\|\widehat{p}(0)-p^*\|^2 \stackrel{\mathrm{def}}{=} \beta^2 \\
&\|Z_{\mathrm{f}}\| \le 1 \qquad \text{and} \qquad \|e\| \le \sqrt{\tfrac{\varepsilon_3}{\varepsilon_2}}\,\|\widehat{p}(0)-p^*\| \\
&\|z_{\mathrm{f}}\| \le \|p^*\| \\
&\frac{I}{\varepsilon_3 + (2-\varepsilon_1)\int_0^T r} \le M(t) \le M(0).
\end{aligned} \tag{409}$$

By taking care of the fact that we are not using v to cancel the term $\frac{\partial V}{\partial p}\,\dot{\widehat{p}}$, we get also:

$$\dot{\widehat{h}} \le -\rho\,e - c\,V + \left(\frac{\partial V}{\partial p} + Z_{\mathrm{f}}\right)\dot{\widehat{p}}. \tag{410}$$

And, by using (409), inequality (380), and the continuity of the function d_2, we have:

$$\left\|\frac{\partial V}{\partial p}\right\| \le d_2(\widehat{p}(t))\,\max\{1\,,\,V\} \tag{411}$$

$$\le k\,(1+V) \tag{412}$$

for some constant k depending only on $\widehat{p}(0)$. Hence, with our choice for ρ and r, we have established:

$$\dot{\widehat{h}} \leq -\rho\, e \,-\, c\, V \,+\, \left(k\,(1+V)+1\right) \left\|\dot{\widehat{p}}\right\| \tag{413}$$

$$\leq -\left(c - k\left\|\dot{\widehat{p}}\right\|\right)\widehat{h} \,+\, \sqrt{r}\,|e| \,+\, c\,|e| \,+\, \left(k\,(1+|e|)+1\right)\left\|\dot{\widehat{p}}\right\| . \tag{414}$$

Hence, with (409), the assumption of Lemma (583) in Appendix B is satisfied with:

$$X \,=\, \widehat{h} \tag{415}$$

and

$$\begin{aligned} \vartheta_1 \,&=\, k\left\|\dot{\widehat{p}}\right\|, & \sigma_1 \,=\, 1,\ S_{11} \,&=\, k\, k_1(p^*,\widehat{p}(0))\\ \varpi_1 \,&=\, \sqrt{r}\,|e| \,+\, c\,|e|\,, & \zeta_1 \,=\, 2,\ S_{21} \,&=\, 2\left(1+c^2\right)\frac{\beta^2}{\varepsilon_1}\\ \varpi_2 \,&=\, \left[k\left(1+\sqrt{\tfrac{\varepsilon_3}{\varepsilon_2}}\,\|\widehat{p}(0)-p^*\|\right)+1\right]\left\|\dot{\widehat{p}}\right\|, & \zeta_2 \,=\, 1,\ S_{22} \,&=\, k_1(p^*,\widehat{p}(0))\,. \end{aligned} \tag{416}$$

¿From here, we conclude the proof exactly as in the previous case. □

In practice, to apply this Proposition, we must be allowed to choose the particular value for the initial condition $\widehat{z}_{\mathrm{f}}(0)$ and a vanishing observer gain K ((263) is assumed). Compare with Proposition (325) where we have no constraint on the initial conditions and the observer gain is not forced to decay to zero. However, Proposition (375) shows that the regressor filtering technique with V as observation function has also the property of providing a solution to the Adaptive Stabilization problem in the case where the Matching Condition (MC) does not hold. Then, besides this question of initial conditions and observer gains, the choice between the two estimation techniques depends on which of the linear growth condition (380) or quadratic growth condition (328) holds.

Example: System (320) Continued **(417)**
For the system (320), let us see if the growth condition (380) holds. From (367), we have:

$$\left\|\frac{\partial V}{\partial p}\right\| \,\leq\, d_2(p)\,\sup\left\{1\,,\,U^{\alpha_2}\right\}\,, \tag{418}$$

where d_2 is given by (368) and α_2 satisfying (369) depends on j and k and is given in Table 2. Hence, for this example (380) is satisfied if we choose $k > 1$ and $j > 1$. This is different from what we obtained in Example (358) where the equation error filtering is to be used. □

5 Estimation Design with an Observation Function h not Directly Related to V

When the observation function h is not directly related to V, the set value maps F and $F^{\dagger}$ defined in (289) do not have any particular properties. Consequently assumption PRS or URS does not give any information on the dynamics of h. Similarly, the estimation does not necessarily provide a better fit of the $\dot{V}$ equation. To overcome

Table 2. $\alpha_2(k,j)$

k \ j	1	2	3	4	5
1	2	5/4	7/6	9/8	11/10
2	5/4	1	1	1	1
3	7/6	11/12	17/18	23/24	29/30
4	9/8	7/8	11/12	15/16	19/20
5	11/10	17/20	9/10	37/40	47/50

this difficulty and guarantee some properties for F and $F^\dagger$, we have to refine our assumptions. Namely, we have to relate V to h. In the global case, i.e., for $\Omega = \mathbb{R}^n$, this yields:

Assumption RBO (Refined Boundedness Observability) (419)
For all positive real numbers α, all compact subsets $\mathcal{K}$ of Π and all vectors $x_0 \in \mathbb{R}^n$, we can find a compact subset Γ of $\mathbb{R}^n$ such that for any C^1 time functions $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$ and any solution $x(t)$ of (44) defined on $[0,T)$, we have the following implication:

$$\|h(x(t),\widehat{p}(t))\| \leq \alpha \text{ and } \widehat{p}(t) \in \mathcal{K}\ \forall t \in [0,T) \quad \to \quad x(t) \in \Gamma\ \forall t \in [0,T)\,. \tag{420}$$

Namely, the boundedness of the full state vector x is "observable" from the boundedness of the "output" function h.

Assumption RURS (Refined Uniform Reduced-Order Stab.) (421)
There exists a positive constant c and two functions:

$$f : \mathbb{R}^k \times \Pi \to \mathbb{R}^k \text{ of class } C^1 \qquad \text{and} \qquad U : \mathbb{R}^k \times \Pi \to \mathbb{R}_+ \text{ of class } C^2\,,$$

with U known *and:*

1. $f(O,p) = 0 \quad \forall p \in \Pi$
2. $U(h,p) = 0 \iff h = 0$
3. $\forall \alpha \geq 0\,,\ \forall \mathcal{K}$ *compact subset of Π, the set*

$$\{h \mid U(h,p) \leq \alpha \text{ and } p \in \mathcal{K}\} \quad \text{is a compact subset of } \mathbb{R}^k\,,$$

such that, for all (x,p,h) in $\mathbb{R}^n \times \Pi \times \mathbb{R}^k$, we have:

$$\begin{aligned} &1.\ f(h(x,p)\,,\,p) = \frac{\partial h}{\partial x}(x,p)\,[a(x,u_{\mathrm{n}}(x,p)) + A(x,u_{\mathrm{n}}(x,p))\,p] \\ &2.\ \frac{\partial U}{\partial h}(h,p)\,f(h,p) \leq -c\,U(h,p)\,. \end{aligned} \tag{422}$$

The meaning of assumption RURS is that, for any fixed vector p in Π, the time derivative of h along the solutions x of (S_p) in closed loop with the control

$$u = u_{\mathrm{n}}(x,p)\,, \tag{423}$$

is simply:

$$\dot{h} \;=\; f(h,p)\,, \tag{424}$$

i.e., this control makes the reduced-order system, consisting of the components of h, autonomous and decoupled from the other components of the full state vector x. Moreover, U is a Lyapunov function for this reduced-order system implying that it admits $0 \in \mathbb{R}^k$ as a globally asymptotically stable equilibrium point.

Assumption RCO (Refined Convergence Observability) (425)
For any bounded C^1 time functions $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$ with $\dot{\widehat{p}}$ also bounded and for any solution $x(t)$ of (44) defined on $[0,+\infty)$, we have the following implication:

$$\left.\begin{array}{l} \lim\limits_{t\to\infty} h(x(t),\widehat{p}(t)) \text{ exists and is zero} \\ \text{and} \quad x(t) \text{ is bounded on } [0,+\infty) \end{array}\right\} \quad\Longrightarrow\quad \lim_{t\to\infty} x(t) \text{ exists and is equal to } \mathcal{E}.$$

Assumption RMC (Refined Matching Condition) (426)
The functions a and A are affine in u, the function U does not depend on p and there exists a known *C^1 function $v(x,p,q,\partial)$ from $\mathbb{R}^n \times \Pi \times \Pi \times \mathbb{R}^l$ to $\mathbb{R}^m$ satisfying:*

$$\frac{\partial h}{\partial p}(x,p)\,\partial + \frac{\partial h}{\partial x}(x,p) v \odot [b(x) + B(x)q] = 0\,. \tag{427}$$

Note the strong restriction on U in assumption RMC.

Example: System (17) Continued (428)
For the system (17) rewritten as (50) in Example (49), the refined assumptions RBO, RCO, RURS and RMC are satisfied if we choose:

$$h(x_1,x_2,p_1,p_2) \;=\; x_1 \quad , \qquad u_{\mathrm{n}}(x_1,x_2,p_1,p_2) \;=\; -\frac{x_2^2+p_1+x_1}{p_2}\;, \tag{429}$$

and:

$$\begin{aligned} U(h,p_1,p_2) &= \left(h+\frac{2}{3}\right)^2 && \text{if} \quad h \le -1 \\ &= \frac{1}{9}\,\exp\left(3\left(1-\frac{1}{h^2}\right)\right) && \text{if} \quad -1 < \; h \; < 1 \\ &= \left(h-\frac{2}{3}\right)^2 && \text{if} \quad 1 \le h\,, \end{aligned} \tag{430}$$

with:

$$f(h,p_1,p_2) \;=\; -h^3\,, \qquad \Pi \;=\; \mathbb{R} \times (\mathbb{R}_+ - \{0\}) \tag{431}$$

and

$$\Gamma \;=\; \left\{x \in \mathbb{R}^2 \mid \|x\| \le \|x(0)\| + \alpha(1+|c_3|)\right\}\,. \tag{432}$$

□

5.1 Equation Error Filtering

Let $\widehat{p}$ be obtained from the equation error filtering technique (230)–(232). If the control is given by:

$$u \;=\; u_{\rm n}(x,\widehat{p})\,, \tag{433}$$

equation (230) gives with the help of (422) in assumption RBO:

$$\dot{\widehat{h}} \;=\; f(\widehat{h}\,,\,\widehat{p}) \;+\; \left[-r(e,x,\widehat{p})\,e + \Big(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p)\Big) + \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}}\right]. \tag{434}$$

If, instead, assumption RMC holds, let $\widehat{p}$ and $\widehat{q}$ be obtained from (234) and use the control:

$$u \;=\; u_{\rm n}(x,\widehat{p}) \;+\; v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right), \tag{435}$$

with v given by (427) in RMC with $\partial = \dot{\widehat{p}}$. Then (434) reduces to:

$$\dot{\widehat{h}} \;=\; f(\widehat{h}\,,\,\widehat{p}) \;+\; \left[-r(e,x,\widehat{p})\,e + \Big(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p)\Big)\right]. \tag{436}$$

In view of assumption RURS, the term enclosed in brackets in both (434) and (436) should be considered as perturbations. To be able to apply techniques a la Total Stability (see Lakshmikantham and Leela [11]), we need to compare the components of this perturbation with the "Lyapunov function" U. This motivates the following assumption:

Assumption GC1 (Growth Conditions 1) (437)
There exist two positive continuous functions d_3, defined on $\Pi \times \mathbb{R}^k$ and d_4, defined on Π and a known *positive real number λ, with $\lambda \le 1$, such that, for all (h,p) in $\mathbb{R}^k \times \Pi$:*

$$\begin{aligned}
&1.\ \left|\frac{\partial U}{\partial h}(h,p)\,(f(h-e\,,\,p) - f(h\,,\,p))\right| \;\le\; d_3(p,e)\,\max\left\{1, U(h,p)^{2-\lambda}\right\}\,\|e\|\\
&2.\ \left\|\frac{\partial U}{\partial h}(h,p)\right\| \;\le\; d_4(p)\,\max\left\{1, U(h,p)^{\lambda}\right\}.
\end{aligned} \tag{438}$$

Moreover, if assumption RMC does not hold, then $\lambda < 1$ and there exist three positive continuous functions d_i, $i = 5,7$ defined on Π and two positive real numbers ω and κ, with $\omega + \kappa \le 2 - \lambda$ such that, for all (x,p) in $\mathbb{R}^n \times \Pi$:

$$\begin{aligned}
&3.\ \left\|\frac{\partial h}{\partial p}(x,p)\right\|\,\left\|\left[\frac{\partial h}{\partial x}(x,p)\,A(x,u_{\rm n}(x,p))\right]^{\rm T}\right\| \le d_5(p)\,\max\{1, U(h(x,p),p)^{2(1-\lambda)}\}\\
&4.\ \left\|\frac{\partial U}{\partial p}(h,p)\right\| \;\le\; d_6(p)\,\max\{1\,,\,U(h,p)^{\omega}\} \qquad \forall\, h \in \mathbb{R}^k\\
&5.\ \left\|\frac{\partial h}{\partial x}(x,p)\,A(x,u_{\rm n}(x,p))\right\| \;\le\; d_7(p)\,\max\{1\,,\,U(h(x,p),p)^{\kappa}\}\,.
\end{aligned} \tag{439}$$

We have:

Proposition **(440)**
Let assumptions Λ-LP, RBO, RURS, ICS and GC1 hold in the global case with

$$A(x,t) \;=\; \frac{\partial h}{\partial x}(x,\widehat{p}(t))\,, \tag{441}$$

and choose:

$$r(e,x,\widehat{p}) \;=\; \left(1 \,+\, U(\widehat{h},\widehat{p})^{(1-\lambda)}\right)^2 . \tag{442}$$

All the corresponding solutions $(x(t),\widehat{p}(t),\widehat{h}(t))$ *of* (S_{p^*})*-(230)-(231) are unique, bounded and well-defined on* $[0,+\infty)$ *and:*

$$\lim_{t\to\infty} h(x(t),\widehat{p}(t)) \;=\; 0\,. \tag{443}$$

It follows that the Adaptive Stabilization problem is solved if assumption RCO holds also.

Proof.
Case: Assumption RMC holds: The system we consider is, with notation (13):

$$\begin{aligned}
\dot{x} &= a\left(x\,,\, u_{\mathrm{n}}(x,\widehat{p}) \,+\, v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right) + A\left(x\,,\, u_{\mathrm{n}}(x,\widehat{p}) \,+\, v(x,\widehat{p},\widehat{q},\dot{\widehat{p}})\right)\, p^* \\
\dot{\widehat{h}} &= -r(e,x,\widehat{p})\, e \,+\, \frac{\partial h}{\partial x}(x,\widehat{p})\,[a_0(x) + A_0(x)\widehat{p} + u_{\mathrm{n}}(x,\widehat{p}) \odot (b(x) + B(x)\widehat{p})] \\
&\quad + \frac{\partial h}{\partial x}(x,\widehat{p})\, v\left(x,\widehat{p},\widehat{q},\dot{\widehat{p}}\right) \odot (b(x) + B(x)\widehat{q}) + \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \\
e &= \widehat{h} \,-\, h(x,\widehat{p}) \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(I,\widehat{p}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\,(A_0(x) + u_{\mathrm{n}}(x,\widehat{p}) \odot B(x))\right]^{\mathrm{T}} e\right)\,,\ \widehat{p}(0) \in \Pi_1 \\
\dot{\widehat{q}} &= \mathrm{Proj}\left(I,\widehat{q}\,, -\left[\frac{\partial h}{\partial x}(x,\widehat{p})\, v(x,\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot B(x)\right]^{\mathrm{T}} e\right)\,,\ \widehat{q}(0) \in \Pi_1\,,
\end{aligned} \tag{444}$$

with $r(e,x,\widehat{p})$ defined by (442) and, with notation (13):

$$\frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}} \,+\, \frac{\partial h}{\partial x}(x,\widehat{p})\, v(x,\widehat{p},\widehat{q},\dot{\widehat{p}}) \odot [b(x) + B(x)\,\widehat{q}\,] \;=\; 0\,. \tag{445}$$

¿From our smoothness assumptions on the various functions and with Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$\left(x,\widehat{p},\widehat{q},\widehat{h}\right) \;\in\; \mathbb{R}^n \times \Pi \times \Pi \times \mathbb{R}^k\,. \tag{446}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{p}(t),\widehat{q}(t),\widehat{h}(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite and satisfying (446). Applying Points 3 and 4 of Lemma (235), we also know that, for all t in $[0,T)$:

$$\begin{aligned}
&\widehat{p}(t) \;\in\; \Pi_1 \quad \text{and} \quad \widehat{q}(t) \;\in\; \Pi_1 \\
&\left\| \begin{matrix} \widehat{p}(t) - p^* \\ \widehat{q}(t) - p^* \end{matrix} \right\|^2 \,+\, \|e(t)\|^2 \,+\, 2\int_0^t r\,\|e\|^2 \;\leq\; \left\| \begin{matrix} \widehat{p}(0) - p^* \\ \widehat{q}(0) - p^* \end{matrix} \right\|^2 \,+\, \|e(0)\|^2 \;\stackrel{\mathrm{def}}{=}\; \beta^2\,.
\end{aligned} \tag{447}$$

Now, from assumption RMC, the function U given by assumption RURS depends on h only. Then, letting:

$$\widehat{U}(t) \,=\, U(\widehat{h}(t))\,, \tag{448}$$

we look at the time derivative of $\widehat{U}$ along the solutions of (444) (see also (436)). ¿From (422) in assumption RURS, (438) in assumption GC1 and (442), we get successively:

$$\begin{aligned}
\dot{\widehat{U}} &\leq -c\,\widehat{U} + \frac{\partial U}{\partial h}(\widehat{h}) \left[-r(e,x,\widehat{p})\,e + \left(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p)\right)\right] \\
&\leq -c\,\widehat{U} + d_4(\widehat{p})\,\max\left\{1,\widehat{U}^{\lambda}\right\} r\,\|e\| \\
&\qquad + d_3(\widehat{p},e)\,\max\left\{1,\widehat{U}^{2-\lambda}\right\}\,\|e\| \\
&\leq -c\,\widehat{U} + d_3(\widehat{p},e)\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\left(1+\widehat{U}\right) \\
&\qquad + d_4(\widehat{p})\,\max\left\{1,\widehat{U}^{\lambda}\right\}\left(1+\widehat{U}^{1-\lambda}\right)^2\,\|e\| \\
&\leq -c\,\widehat{U} + \left[(d_3(\widehat{p},e) + 2\,d_4(\widehat{p}))\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]\left(1+\widehat{U}\right) \\
&\leq -\left[c - k\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]\widehat{U} + k\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\,,
\end{aligned} \tag{449}$$

where the constant k depends only on the initial conditions and satisfies:

$$d_3(\widehat{p}(t),e(t)) + 2\,d_4(\widehat{p}(t)) \,\leq\, k \qquad \forall\, t \,\in\, [0,T)\,. \tag{450}$$

Such an inequality holds since we have (447) and the functions d_3 and d_4 are continuous. With (442), inequality (449) implies that the assumption of Lemma (583) in Appendix B is satisfied with:

$$X \,=\, \widehat{U} \tag{451}$$

and:

$$\begin{aligned}
\vartheta_1 \,&=\, k\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\,,\ \sigma_1 \,=\, 2,\, S_{11} \,=\, k^2\,\frac{\beta^2}{2} \\
\varpi_1 \,&=\, k\,\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\,,\, \zeta_1 \,=\, 2,\ S_{21} \,=\, k^2\,\frac{\beta^2}{2}\ .
\end{aligned} \tag{452}$$

With the property 3 of U in assumption RURS, it follows that there exists a constant Υ depending only on $e(0)$, $\widehat{p}(0)$ and $\widehat{q}(0)$ such that, for all t in $[0,T)$, we have:

$$\left\|\widehat{h}(t)\right\| \,\leq\, \Upsilon\,. \tag{453}$$

Then, with (447), this implies the existence of a constant α depending only on $e(0)$, $\widehat{p}(0)$ and $\widehat{q}(0)$ such that, for all t in $[0,T)$:

$$\begin{aligned}
\|h(x(t),\widehat{p}(t))\| &\leq \left\|\widehat{h}(t)\right\| \,+\, \|e(t)\| & (454)\\
&\leq \alpha\,, & (455)
\end{aligned}$$

and we also know from (447) that $\widehat{p}(t) \in \mathcal{K}$ and $\widehat{q}(t) \in \mathcal{K}$, where $\mathcal{K}$ is the following compact subset of Π:

$$\mathcal{K} \,=\, \{p \mid \|p-p^*\| \,\leq\, \beta\,\} \bigcap \Pi_1\,. \tag{456}$$

Then, from assumption RBO, there exists of a compact subset Γ of $\mathbb{R}^n$ such that:

$$x(t) \,\in\, \Gamma \quad \forall\, t \,\in\, [0,T)\,. \tag{457}$$

With (447) and (453), we have established that the solution remains in a compact subset of the open set defined by (446). This implies by contradiction that $T = +\infty$ and that $x(t)$, $\widehat{p}(t)$, $u(t)$ and $\dot{\widehat{p}}(t)$ are bounded on $[0, +\infty)$.

Then, using the second conclusion of Lemma (583) and the properties of the function U, we have:

$$\lim_{t \to +\infty} \widehat{h}(t) \;=\; 0\,. \tag{458}$$

Also, from (229) and the fact that the solution is bounded, we deduce that $\dot{e}$ is bounded. Since, from (447), e is in $L^2([0, +\infty))$, we have also:

$$\lim_{t \to +\infty} e(t) \;=\; 0\,. \tag{459}$$

This yields:

$$\lim_{t \to +\infty} h(x(t), \widehat{p}(t)) \;=\; \lim_{t \to +\infty} \widehat{h}(t) \;-\; \lim_{t \to +\infty} e(t) \;=\; 0\,. \tag{460}$$

With assumption RCO this implies finally:

$$\lim_{t \to +\infty} x(t) \;=\; \mathcal{E}\,. \tag{461}$$

Case: Assumption RMC does not hold: The system we consider is:

$$\begin{aligned}
\dot{x} &= a\,(x\,,\,u_{\mathrm{n}}(x, \widehat{p})) + A\,(x\,,\,u_{\mathrm{n}}(x, \widehat{p}))\;p^* \\
\dot{\widehat{h}} &= -r(e, x, \widehat{p})\,e \,+\, \frac{\partial V}{\partial x}(x, \widehat{p})\,A(x, u_{\mathrm{n}}(x, \widehat{p}))\,\widehat{p} \,+\, \frac{\partial V}{\partial x}(x, \widehat{p})\,a(x, u_{\mathrm{n}}(x, \widehat{p})) \\
&\qquad + \frac{\partial V}{\partial p}(x, \widehat{p})\,\dot{\widehat{p}} \\
e &= \widehat{h} \,-\, V(x, \widehat{p}) \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(I\,,\,\widehat{p}\,,\,-\left[\frac{\partial h}{\partial x}(x, \widehat{p})\,A(x, u_{\mathrm{n}}(x, \widehat{p}))\right]^{\mathrm{T}} e\right), \qquad \widehat{p}(0) \,\in\, \mathit{\Pi}_1\,,
\end{aligned} \tag{462}$$

with $r(e, x, \widehat{p})$ defined by (442). ¿From our smoothness assumptions on the various functions and from Point 1 of Lemma (103), this system has a locally Lipschitz-continuous right-hand side in $\mathbb{R}^n \times \mathit{\Pi} \times \mathbb{R}^k$. It follows that, for any initial condition in this open set, there exists a unique solution $(x(t), \widehat{p}(t), \widehat{h}(t))$, defined on a right maximal interval $[0, T)$, with T possibly infinite.

Applying Points 1 and 2 of Lemma (235), we also know that, for all t in $[0, T)$

$$\begin{aligned}
&\widehat{p}(t) \,\in\, \mathit{\Pi}_1 \\
&\|\widehat{p}(t) - p^*\|^2 \,+\, \|e(t)\|^2 \,+\, 2\int_0^t r\,\|e\|^2 \;\leq\; \|\widehat{p}(0) - p^*\|^2 \,+\, \|e(0)\|^2 \;\stackrel{\mathrm{def}}{=}\; \beta^2\,.
\end{aligned} \tag{463}$$

Then, as in the previous case, we study the evolution of the time derivative of $\widehat{U}$ along the solutions of (462) (see also (434)), with:

$$\widehat{U}(t) \;=\; U(\widehat{h}(t), \widehat{p}(t))\,. \tag{464}$$

¿From (422) in assumption RURS, Point 2 of Lemma (103), (438) and (439) in assumption GC1, (442), and the expression of $\dot{\widehat{p}}$ in (462), we get successively:

$$\begin{aligned}\dot{\widehat{U}} &\leq -c\,\widehat{U} + \frac{\partial U}{\partial h}(\widehat{h},\widehat{p})\left[-r\,e + \left(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p)\right) + \frac{\partial h}{\partial p}(x,\widehat{p})\dot{\widehat{p}}\right] \\ &\quad + \frac{\partial U}{\partial p}(\widehat{h},\widehat{p})\dot{\widehat{p}} \\ &\leq -c\,\widehat{U} \\ &\quad + \left\|\frac{\partial U}{\partial h}(\widehat{h},\widehat{p})\right\| \left[r\,\|e\| + \left\|\frac{\partial h}{\partial p}(\widehat{h},\widehat{p})\right\| \left\|\left[\frac{\partial h}{\partial x}(x,\widehat{p})\,A(x,u_{\mathrm{n}}(x,\widehat{p}))\right]^{\mathrm{T}}\right\| \|e\|\right] \\ &\quad + \left|\frac{\partial U}{\partial h}(\widehat{h},\widehat{p})\left(f(\widehat{h}-e\,,\widehat{p}) - f(\widehat{h},p)\right)\right| \\ &\quad + \left\|\frac{\partial U}{\partial p}(\widehat{h},\widehat{p})\right\| \left\|\left[\frac{\partial h}{\partial x}(x,\widehat{p})\,A(x,u_{\mathrm{n}}(x,\widehat{p}))\right]^{\mathrm{T}}\right\| \|e\| \\ &\leq -c\,\widehat{U} \\ &\quad + d_4(\widehat{p})\,\max\left\{1,\widehat{U}^{\lambda}\right\}\left[r\,\|e\| + d_5(\widehat{p})\,\max\{1\,,\,U(h(x,\widehat{p}),\widehat{p})^{2(1-\lambda)}\}\,\|e\|\right] \\ &\quad + d_3(\widehat{p},e)\,\max\left\{1,\widehat{U}^{2-\lambda}\right\}\,\|e\| \\ &\quad + d_6(\widehat{p})\,\max\{1\,,\,\widehat{U}^{\omega}\}\,d_7(\widehat{p})\,\max\{1\,,\,U(h(x,\widehat{p}),\widehat{p})^{\kappa}\}\,\|e\|\,. \end{aligned} \tag{465}$$

A difficulty appearing in this inequality and which we have not encountered yet is the distinction between $\widehat{U} = U(\widehat{h},\widehat{p})$ and $U(h,\widehat{p}) = U(\widehat{h}-e\,,\,\widehat{p})$. As proved in Appendix C, thanks to point 2 in (438) of assumption GC1, these two quantities are related by:

$$\max\{1\,,\,U(h(x,\widehat{p}),\widehat{p})^{\gamma}\} \;\leq\; \delta\left[\max\{1\,,\,\widehat{U}^{\gamma}\} + \max\{1\,,\,\widehat{U}^{\lambda\gamma}\}\left(\|e\|^{\gamma} + \|e\|^{\frac{\gamma}{1-\lambda}}\right)\right], \tag{466}$$

where γ is any positive real number and $\delta \geq 1$ depends only on $d_4(\widehat{p})$. Since the functions d_i, $i=3,7$ are continuous and e and $\widehat{p}$ are bounded from (463), there exists a constant k depending only on the initial conditions such that inequality (465) yields:

$$\begin{aligned}\dot{\widehat{U}} &\leq -c\,\widehat{U} \\ &\quad + k\,\max\left\{1,\widehat{U}^{\lambda}\right\}\left(1+\widehat{U}^{1-\lambda}\right)^{2}\|e\| \\ &\quad + k\,\max\left\{1\,,\,\widehat{U}^{\lambda}\right\} \\ &\qquad \times\left[\max\left\{1,\widehat{U}^{2(1-\lambda)}\right\} + \max\left\{1,\widehat{U}^{2\lambda(1-\lambda)}\right\}\left(\|e\|^{2(1-\lambda)} + \|e\|^{2}\right)\right]\|e\| \\ &\quad + k\,\max\left\{1,\widehat{U}^{2-\lambda}\right\}\|e\| \\ &\quad + k\,\max\left\{1,\widehat{U}^{\omega}\right\} \\ &\qquad \times\left[\max\left\{1,\widehat{U}^{\kappa}\right\} + \max\left\{1,\widehat{U}^{\lambda\kappa}\right\}\left(\|e\|^{\kappa} + \|e\|^{\frac{\kappa}{1-\lambda}}\right)\right]\|e\|\,. \end{aligned} \tag{467}$$

Since assumption GC1 gives

$$\omega + \kappa \leq 2-\lambda\,, \tag{468}$$

we have:

$$\max\{1\,,\,\widehat{U}^{\omega}\}\,\max\{1\,,\,\widehat{U}^{\kappa}\} \;\leq\; \left(1+\widehat{U}\right)\left(1+\widehat{U}\right)^{1-\lambda}. \tag{469}$$

Hence, (467) can be simplified in:

$$\begin{aligned}\dot{\widehat{U}} \leq & -c\,\widehat{U} \\ & + 5\,k\,\left(1+\widehat{U}\right)\left(1+\widehat{U}^{1-\lambda}\right)\|e\| \\ & + k\,\max\{1\,,\,\widehat{U}^{\lambda}\}\left[\max\{1\,,\,\widehat{U}^{2\lambda(1-\lambda)}\}\left(\|e\|^{2(1-\lambda)+1} + \|e\|^{3}\right)\right] \stackrel{\text{def}}{=} \text{(a)} \\ & + k\,\max\{1\,,\,\widehat{U}^{\omega}\}\left[\max\{1\,,\,\widehat{U}^{\lambda\kappa}\}\left(\|e\|^{\kappa+1} + \|e\|^{1+\frac{\kappa}{1-\lambda}}\right)\right] \stackrel{\text{def}}{=} \text{(b)}\,.\end{aligned} \tag{470}$$

Let us now show that the terms (a) and (b) defined in this inequality can be bounded from above by terms of the form:

$$\left(1+\widehat{U}\right)\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{\gamma}\,,$$

with $0 < \gamma \leq 2$.

(a) If $2\lambda - 1 > 0$, and since

$$\lambda + 2\,\lambda\,(1-\lambda) \;=\; 1 \;+\; (2\lambda - 1)\,(1-\lambda)\,, \tag{471}$$

we get:

$$\text{(a)} \;\leq\; k\left(1+\widehat{U}\right)\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{2\lambda-1}\left(\|e\|^{4(1-\lambda)} + \|e\|^{4-2\lambda}\right), \tag{472}$$

where, knowing from (463) that $\|e\|$ is bounded, the same holds for the last term between parentheses, since we have positive powers.
If $2\lambda - 1 \leq 0$, and since

$$\lambda + 2\,\lambda\,(1-\lambda) \;\leq\; 2\,(1-\lambda)\,, \tag{473}$$

we get:

$$\text{(a)} \leq k\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{2}\left(\|e\|^{1-2\lambda} + \|e\|\right) \tag{474}$$

$$\leq k\left(1+\widehat{U}\right)\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{2}\left(\|e\|^{1-2\lambda} + \|e\|\right). \tag{475}$$

(b) If $\kappa < 1$, and since

$$\omega \;+\; \lambda\,\kappa \;\leq\; 1 \;+\; (1-\kappa)\,(1-\lambda)\,, \tag{476}$$

we get:

$$\text{(b)} \;\leq\; k\left(1+\widehat{U}\right)\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{1-\kappa}\left(\|e\|^{2\kappa} + \|e\|^{\frac{\kappa}{1-\lambda}+\kappa}\right). \tag{477}$$

If $\kappa \geq 1$, and since

$$\omega \;+\; \lambda\,\kappa \;\leq\; 1 \;+\; 2\,(1-\lambda)\,, \tag{478}$$

we get:

$$\text{(b)} \;\leq\; k\left(1+\widehat{U}\right)\left[\left(1+\widehat{U}^{1-\lambda}\right)\|e\|\right]^{2}\left(\|e\|^{\kappa-1} + \|e\|^{\frac{\kappa}{1-\lambda}-1}\right). \tag{479}$$

With these inequalities, and the expression (442) for r, (470) yields an inequality we write formally as:

$$\dot{\widehat{U}} \leq -c\widehat{U} + 5k\left(1+\widehat{U}\right)\sqrt{r}\|e\| + \left(1+\widehat{U}\right)\left[\overline{k}_{\mathrm{a}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}} + \overline{k}_{\mathrm{b}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{b}}}\right], \tag{480}$$

where the last term represents the sum (a)+(b), with $\overline{k}_{\mathrm{a}}$ and $\overline{k}_{\mathrm{b}}$ two constants depending only on the initial conditions, and

$$0 < \gamma_{\mathrm{a}} \leq 2, \qquad 0 < \gamma_{\mathrm{b}} \leq 2. \tag{481}$$

This inequality can be rewritten as:

$$\dot{\widehat{U}} \leq -\left(c - 5k\sqrt{r}\|e\| - \left[\overline{k}_{\mathrm{a}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}} + \overline{k}_{\mathrm{b}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{b}}}\right]\right)\widehat{U} + 5k\sqrt{r}\|e\| + \left[\overline{k}_{\mathrm{a}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}} + \overline{k}_{\mathrm{b}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{b}}}\right]. \tag{482}$$

We may now apply Lemma (583) in Appendix B with

$$X = \widehat{U} \tag{483}$$

and

$$\begin{aligned}
\vartheta_1 &= 5k\sqrt{r}\|e\|, & \sigma_1 &= 2, & S_{11} &= 25k^2\frac{\beta^2}{2} \\
\vartheta_2 &= \overline{k}_{\mathrm{a}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}}, & \sigma_2 &= \frac{2}{\gamma_{\mathrm{a}}}, & S_{12} &= \overline{k}_{\mathrm{a}}^{\,2}\frac{\beta^2}{2} \\
\vartheta_3 &= \overline{k}_{\mathrm{b}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{b}}}, & \sigma_3 &= \frac{2}{\gamma_{\mathrm{b}}}, & S_{13} &= \overline{k}_{\mathrm{b}}^{\,2}\frac{\beta^2}{2} \\
\varpi_1 &= 5k\sqrt{r}\|e\|, & \zeta_1 &= 2, & S_{21} &= 25k^2\frac{\beta^2}{2} \\
\varpi_2 &= \overline{k}_{\mathrm{a}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}}, & \zeta_2 &= \frac{2}{\gamma_{\mathrm{a}}}, & S_{22} &= \overline{k}_{\mathrm{a}}^{\,2}\frac{\beta^2}{2} \\
\varpi_3 &= \overline{k}_{\mathrm{b}}\left(\sqrt{r}\|e\|\right)^{\gamma_{\mathrm{a}}}, & \zeta_3 &= \frac{2}{\gamma_{\mathrm{b}}}, & S_{23} &= \overline{k}_{\mathrm{b}}^{\,2}\frac{\beta^2}{2}.
\end{aligned} \tag{484}$$

¿From here, we conclude the proof exactly as in the previous case. □

This proposition shows that, in fact, the observation h cannot be chosen independently of V. The refined assumptions RCO and RURS specify the link between these two functions with U playing the role of V. However, not taking $h = V$ creates some difficulties. Even when the Refined Matching Condition (RMC) is met, some extra conditions are required – the increment condition in point 1 and the growth condition in point 2 of assumption GC1. Nevertheless, as with the other estimation designs, we may have a solution to the Adaptive Stabilization problem when assumption RMC does not hold.

Corollary [Campion and Bastin[2]] (485)
Let, in equation (S_p)*, the functions* a *and* A *be* known *and affine in* u *and let* Π *be an open subset of* $\mathbb{R}^l$ *which satisfies assumption ICS. Assume there exist three* known *functions:*

$h \;: \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ *of class* C^2*, a diffeomorphism for each* p*,*
$u_{\mathrm{n}} : \mathbb{R}^n \times \Pi \to \mathbb{R}^m$ *of class* C^1*, and*
$v \;: \mathbb{R}^n \times \Pi \times \Pi \times \mathbb{R}^l \to \mathbb{R}^m$ *of class* C^1*,*

such that:

1. *h satisfies assumption RCO,*
2. *for all (x, p, q, ∂) in $\mathbb{R}^n \times \Pi \times \Pi \times \mathbb{R}^l$, we have:*

$$\frac{\partial h}{\partial p}(x,p)\,\partial + \frac{\partial h}{\partial x}(x,p)\,v \odot [b(x) + B(x)\,q] \;=\; 0\,, \tag{486}$$

3. *and by letting*

$$\varphi \;=\; h(x,p)\,, \tag{487}$$

the time derivative of φ along the solutions of (S_p) with $u = u_{\mathrm{n}}$ satisfies:

$$\dot{\varphi} \;=\; C\,\varphi\,, \tag{488}$$

where C is an $n \times n$ matrix satisfying:

$$P\,C + C^{\mathrm{T}} P \;=\; -I\,, \tag{489}$$

with P a symmetric positive definite matrix.

Under these conditions, the Adaptive Stabilization problem is solved by the dynamic controller consisting of (234) and:

$$u \;=\; u_{\mathrm{n}}(x, \widehat{p}) + v\left(x, \widehat{p}, \widehat{q}, \dot{\widehat{p}}\right). \tag{490}$$

Proof. Let us first notice that, the functions a, A and h being known, assumption Λ-LP holds with:

$$\Lambda(x,t) \;=\; \frac{\partial h}{\partial x}(x, \widehat{p}(t))\,. \tag{491}$$

Then, we define two functions U and f as follows:

$$U(h) \;=\; h^{\mathrm{T}} P\, h \tag{492}$$

and

$$f(h\,,\,p) \;=\; C\,h\,. \tag{493}$$

¿From the assumptions, conditions Λ-LP, RURS and RMC are satisfied.

Now, define the function F by:

$$\begin{array}{rcl} F : \mathbb{R}^n \times \Pi & \to & \mathbb{R}^n \times \Pi\,. \\ (x,p) & & (h(x,p)\,,\,p) \end{array} \tag{494}$$

This function is a diffeomorphism. Hence, for all compact subsets $\mathcal{K}$ of Π and all positive real numbers α, the set:

$$F^{-1}\left\{(\varphi, p) \mid p \in \mathcal{K} \quad \text{and} \quad \|\varphi\| \,\le\, \alpha\right\}$$

is a compact subset of $\Omega \times \Pi$, and therefore its projection:

$$\Gamma_{\alpha,\mathcal{K}} \;=\; \{x \mid \exists\, p \,\in\, \mathcal{K} \,:\, \|h(x,p)\| \,\le\, \alpha\} \tag{495}$$

is a compact subset of $\mathbb{R}^n$. It follows that assumption RBO holds.

Finally, let us check that (438) in assumption GC1 is satisfied. We have:

$$\left\|\frac{\partial U}{\partial h}(h,p)\right\| = \left\|h^{\mathrm{T}} P\right\| \tag{496}$$

$$\le \sqrt{\lambda_{\max}\{P\}}\, U(h)^{\frac{1}{2}}\,, \tag{497}$$

and also:

$$\left|\frac{\partial U}{\partial h}(h,p)\,(f(h-e\,,\,p)-f(h\,,\,p))\right| = \left|h^{\mathrm{T}}P\,Ce\right| \tag{498}$$

$$\leq U(h)^{\frac{1}{2}}\,\sqrt{e^{\mathrm{T}}C^{\mathrm{T}}P\,C\,e} \tag{499}$$

$$\leq \sqrt{\lambda_{\max}\left\{C^{\mathrm{T}}P\,C\right\}}\,U(h)^{\frac{1}{2}}\,\|e\|\,. \tag{500}$$

Hence, (438) holds with any λ satisfying:

$$\frac{1}{2}\,\leq\,\lambda\,\leq\,1\,. \tag{501}$$

So we choose $\lambda = 1$ in order to simplify the expression of r in (442). □

Note that in Corollary (485) the Matching Condition (486) can be replaced by Points 3 and 5 of assumption GC1, namely:

There exist two positive continuous functions d_5 and d_7 defined on Π such that, for all (x,p) in $\mathbb{R}^n\times\Pi$:

$$\begin{aligned}&\left\|\frac{\partial h}{\partial p}(x,p)\right\|\cdot\left\|\left[\frac{\partial h}{\partial x}(x,p)\,A(x,u_{\mathrm{n}}(x,p))\right]^{\mathrm{T}}\right\|\leq d_5(p)\,\max\{1\,,\,\|h(x,p)\|^2\}\\&\left\|\frac{\partial h}{\partial x}(x,p)\,A(x,u_{\mathrm{n}}(x,p))\right\|\leq d_7(p)\,\max\{1\,,\,\|h(x,p)\|^3\}\,.\end{aligned} \tag{502}$$

5.2 Regressor Filtering

When the regressor filtering technique (258)–(261) is applied, by letting:

$$\widehat{h}\,=\,h(x,\widehat{p})\,+\,e=\,Z_{\mathrm{f}}\,\widehat{p}\,+\,\widehat{z}_{\mathrm{f}}\,, \tag{503}$$

and using (259) and (261), we get:

$$\dot{\widehat{h}}\,=\,f(\widehat{h}\,,\,\widehat{p})\,+\,\left[-\rho\,e+\left(f(\widehat{h}-e\,,\,\widehat{p})-f(\widehat{h},\widehat{p})\right)+\left(\frac{\partial h}{\partial p}+Z_{\mathrm{f}}\right)\dot{\widehat{p}}\right], \tag{504}$$

if the control u is:

$$u\,=\,u_{\mathrm{n}}(x,\widehat{p})\,. \tag{505}$$

And we get:

$$\dot{\widehat{h}}\,=\,f(\widehat{h}\,,\,\widehat{p})\,+\,\left[-\rho\,e+\left(f(\widehat{h}-e\,,\,\widehat{p})-f(\widehat{h},\widehat{p})\right)+Z_{\mathrm{f}}\,\dot{\widehat{p}}\right], \tag{506}$$

if assumption RMC holds, A does not depend on u and we use:

$$u\,=\,u_{\mathrm{n}}(x,\widehat{p})\,+\,v(x,\widehat{p},\dot{\widehat{p}})\,, \tag{507}$$

where, with notation (13):

$$\frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}}+\frac{\partial h}{\partial x}(x,\widehat{p})v(x,\widehat{p},\dot{\widehat{p}})\odot b(x)\,=\,0\,. \tag{508}$$

Compared with the equation error filtering case, we have the extra term $Z_{\mathrm{f}}\dot{\widehat{p}}$ in both (504) and (506). However, we know that, by choosing ρ appropriately, Z_{f} is bounded. And, by using a vanishing observation gain, $\dot{\widehat{p}}$ is absolutely integrable. This latter property being different from what we have in the error filtering case, different growth conditions are needed:

Assumption GC2 (Growth Conditions 2) (509)
There exist three positive continuous functions d_i , $i = 8, 10$, with d_8 defined on $\Pi \times \mathbb{R}^k$ and d_9 and d_{10} defined on Π, two positive real numbers λ and κ, with $\lambda < 1$ and κ known*, such that, for all (φ, x, p) in $\mathbb{R}^k \times \mathbb{R}^n \times \Pi$:*

$$
\begin{aligned}
&1.\ \left|\frac{\partial U}{\partial h}(\varphi,p)\left(f(\varphi-e\,,\,p)-f(\varphi\,,\,p)\right)\right| \,\le\, d_8(p,e)\,\max\left\{1, U(\varphi,p)^{\kappa+\lambda}\right\}\,\|e\|\\
&2.\ \left\|\frac{\partial U}{\partial h}(\varphi,p)\right\| \,\le\, d_9(p)\,\max\left\{1, U(\varphi,p)^{\lambda}\right\}\\
&3.\ \left\|\frac{\partial h}{\partial x}(x,p)\,A(x,u_{\mathrm{n}}(x,p))\right\| \,\le\, d_{10}(p)\,\max\left\{1\,,\,U(h(x,p),p)^{\kappa}\right\}\,.
\end{aligned}
\tag{510}
$$

Moreover, if assumption RMC does not hold, there exist two positive continuous functions d_{11} and d_{12}, defined on Π such that, for all (φ, x, p) in $\mathbb{R}^k \times \mathbb{R}^n \times \Pi$:

$$
\begin{aligned}
&4.\ \left\|\frac{\partial h}{\partial p}(x,p)\right\| \,\le\, d_{11}(p)\,\max\left\{1\,,\,U(h(x,p),p)^{(1-\lambda)}\right\}\\
&5.\ \left\|\frac{\partial U}{\partial p}(\varphi,p)\right\| \,\le\, d_{12}(p)\,\max\left\{1\,,\,U(\varphi,p)\right\}\,.
\end{aligned}
\tag{511}
$$

Note that there is no constraint on κ besides its existence. We have:

Proposition (512)
Let assumptions Λ-LP, RBO, RURS, ICS and GC2 hold in the global case with:

$$\Lambda(x,t) \,=\, \frac{\partial h}{\partial x}(x,\widehat{p}(t))\,. \tag{513}$$

Choose:

$$r(x,\widehat{p}) \,=\, \left(1 \,+\, U(\widehat{h},\widehat{p})^{\kappa}\right)^2 \tag{514}$$

$$\rho(x,\widehat{p},u) \,=\, 1 + \left\|\frac{\partial h}{\partial x}(x,p)\,A(x,u)\right\|\,, \tag{515}$$

and, in (262):

$$\dot{M} \,=\, G\,(M\,,\,Z_{\mathrm{f}}\,,\,r)\,, \qquad \varepsilon_3\,M(0) \,>\, I\,, \tag{516}$$

where G is a negative symmetric matrix, depending locally-Lipschitz-continuously on its arguments and satisfying (see (262) and (263)):

$$-\varepsilon_4\,M\,Z_{\mathrm{f}}^{\mathrm{T}}\,Z_{\mathrm{f}}\,M\,r \,\ge\, G \,\ge\, -(2-\varepsilon_1)\,M\,Z_{\mathrm{f}}^{\mathrm{T}}\,Z_{\mathrm{f}}\,M\,r\,. \tag{517}$$

Assume that A does not depend on u when u is chosen to depend on $\dot{\widehat{p}}$. Under these conditions, all the solutions $(x(t), \widehat{p}(t), \widehat{z}_{\mathrm{f}}(t), Z_{\mathrm{f}}(t)), M(t)$ of (S_{p^*}), (259), (261), (262), (516) are well-defined on $[0, +\infty)$, unique, bounded and:

$$\lim_{t\to\infty} h(x(t),\widehat{p}(t)) \,=\, 0\,. \tag{518}$$

It follows that the Adaptive Stabilization problem is solved if assumption RCO holds also.

Proposition (512) generalizes a result established by Pomet and Praly [22] who choose the state vector x as the observation function h.

Proof.

Case: Assumption MC holds: In this case, we assume also that $A(x,u)$ does not depend on u, i.e., with notation (13), we have:

$$A(x,u) \;=\; A_0(x)\,. \tag{519}$$

The system we consider is:

$$\begin{aligned}
\dot{x} &= a\,(x\,,\,u) + A_0(x)\,p^* \\
\dot{\widehat{z}}_{\mathrm{f}} &= \rho(x,\widehat{p},u)\,z_{\mathrm{f}} + \frac{\partial h}{\partial x}(x,\widehat{p})\,a(x,u) \;+\; \frac{\partial h}{\partial p}(x,\widehat{p})\,\dot{\widehat{p}}\,, \quad \widehat{z}_{\mathrm{f}}(0) \;=\; h(x(0),\widehat{p}(0)) \\
z_{\mathrm{f}} &= h(x,\widehat{p}) \;-\; \widehat{z}_{\mathrm{f}} \\
\dot{Z}_{\mathrm{f}} &= -\rho(x,\widehat{p})\,Z_{\mathrm{f}} + \frac{\partial h}{\partial x}(x,\widehat{p})\,A_0(x)\,, \quad Z_{\mathrm{f}}(0) \;=\; 0 \\
e &= Z_{\mathrm{f}}\,\widehat{p} \;-\; z_{\mathrm{f}} \\
\dot{\widehat{p}} &= \mathrm{Proj}\left(M\,,\,\widehat{p}\,,\,-M\,Z_{\mathrm{f}}^{\mathrm{T}}\,r(x,\widehat{p})\,e\right), \quad \widehat{p}(0) \;\in\; \Pi_1 \\
\dot{M} &= G\,(M\,,\,Z_{\mathrm{f}}\,,\,r(x,\widehat{p}))\;, \quad M(0) \;>\; 0\,,
\end{aligned} \tag{520}$$

with r given by (514), ρ given by (515) and:

$$u \;=\; u_{\mathrm{n}}(x,\widehat{p}) + v(x,\widehat{p},\dot{\widehat{p}})\,, \tag{521}$$

where v satisfies (515). Since r does not depend on u, $\dot{\widehat{p}}$ is explicitly defined. Then, from our smoothness assumptions on the various functions and with Lemma (103), this system has a locally Lipschitz-continuous right-hand side in the open set defined by:

$$(x,\widehat{z}_{\mathrm{f}},Z_{\mathrm{f}},\widehat{p},M) \;\in\; \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{M}_{kl}(\mathbb{R}) \times \Pi \times \mathcal{M}\,. \tag{522}$$

It follows that, for any initial condition in this open set, there exists a unique solution $(x(t),\widehat{z}_{\mathrm{f}}(t),Z_{\mathrm{f}}(t),\widehat{p}(t),M(t))$, defined on a right maximal interval $[0,T)$, with T possibly infinite, and satisfying (522). Then, as in the proof of Proposition (375), we have, for all t in $[0,T)$:

$$\begin{aligned}
&\widehat{p}(t) \;\in\; \Pi_1 \\
&\varepsilon_2\,\|\widehat{p}(t) - p^*\|^2 \;+\; \varepsilon_1 \int_0^t r\;\|e\|^2 \;\leq\; \varepsilon_3\,\|\widehat{p}(0) - p^*\|^2 \;\stackrel{\mathrm{def}}{=}\; \beta^2 \\
&\|Z_{\mathrm{f}}\| \;\leq\; 1 \qquad \text{and} \qquad \|e\| \;\leq\; \sqrt{\frac{\varepsilon_3}{\varepsilon_2}}\,\|\widehat{p}(0) - p^*\| \\
&\|z_{\mathrm{f}}\| \;\leq\; \|p^*\| \\
&\frac{I}{\varepsilon_3 + (2-\varepsilon_1)\int_0^T r} \;\leq\; M(t) \;\leq\; M(0)\,.
\end{aligned} \tag{523}$$

Now, as in the proof of Proposition (440), we look at the time derivative along the solutions of (520) (see also (506)) of $\widehat{U}$ defined in (448). ¿From (510) in assumption GC2, (422) in assumption RURS, (514), (515), and (523), we get successively:

$$\begin{aligned}\dot{\widehat{U}} &\leq -c\,\widehat{U} + \frac{\partial U}{\partial h}(\widehat{h}) \left[-\rho(x,\widehat{p},u)\,e + \left(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p)\right) + Z_{\mathrm{f}}\,\dot{\widehat{p}}\right] \\ &\leq -c\,\widehat{U} \\ &\quad + d_9(\widehat{p})d_{10}(\widehat{p})\,\max\left\{1,\widehat{U}^{\lambda}\right\}\left(1+\max\left\{1\,,\,U(h(x,\widehat{p}))^{\kappa}\right\}\right)\|e\| \\ &\quad + d_8(\widehat{p},e)\,\max\left\{1,\widehat{U}^{\kappa+\lambda}\right\}\|e\| \\ &\quad + d_9(\widehat{p})\,\max\left\{1\,,\,\widehat{U}^{\lambda}\right\}\left\|\dot{\widehat{p}}\right\|\,. \end{aligned} \tag{524}$$

And, with Appendix C, we have the following inequality:

$$\begin{aligned}\max\left\{1\,,\,U(h(x,\widehat{p}),\widehat{p})^{\kappa}\right\} &\leq \delta\,\max\left\{1\,,\,\widehat{U}^{\kappa}\right\} \\ &\quad + \delta\left[\max\left\{1\,,\,\widehat{U}^{\lambda\kappa}\right\}\left(\|e\|^{\kappa}+\|e\|^{\frac{\kappa}{1-\lambda}}\right)\right]\end{aligned} \tag{525}$$

for some constant $\delta \geq 1$ depending only on $d_9(\widehat{p})$. Since the functions d_i, $i = 8, 10$ are continuous and e and $\widehat{p}$ are bounded from (523), there exists a constant k depending only on the initial conditions such that:

$$\begin{aligned}\dot{\widehat{U}} &\leq -c\,\widehat{U} \\ &\quad + k\left(1+\widehat{U}^{\lambda}\right)\left[\|e\| + 2\left(1+\widehat{U}^{\kappa}\right)\|e\|\right] \\ &\quad + k\left(1+\widehat{U}^{\lambda}\right)\left[\left(1+\widehat{U}^{\kappa}\right)\|e\|\right]^{\lambda}\left(\|e\|^{1+\kappa-\lambda} + \|e\|^{\frac{\kappa}{1-\lambda}+1-\lambda}\right) \\ &\quad + k\left(1+\widehat{U}^{\lambda}\right)\left\|\dot{\widehat{p}}\right\|\,. \end{aligned} \tag{526}$$

Finally, since r satisfies (514), e is bounded, and from GC2 we have $\lambda < 1$, we get more simply:

$$\begin{aligned}\dot{\widehat{U}} &\leq -\left[c - \left(k\|e\| + 2k\sqrt{r}\|e\| + \overline{k}\left(\sqrt{r}\|e\|\right)^{\lambda} + k\left\|\dot{\widehat{p}}\right\|\right)\right]\widehat{U} \\ &\qquad + 2k\sqrt{r}\|e\| + \overline{k}\left(\sqrt{r}\|e\|\right)^{\lambda} + k\left\|\dot{\widehat{p}}\right\|\,. \end{aligned} \tag{527}$$

We may now apply Lemma (583) in Appendix B with:

$$X = \widehat{U} \tag{528}$$

and:

$$\begin{array}{lll} \vartheta_1 = k\|e\| + 2k\sqrt{r}\|e\|\,, & \sigma_1 = 2, & S_{11} = 10\,k^2\,\dfrac{\beta^2}{\varepsilon_1} \\ \vartheta_2 = \overline{k}\left(\sqrt{r}\|e\|\right)^{\lambda}, & \sigma_2 = \dfrac{2}{\lambda}\,, & S_{12} = \overline{k}^2\,\dfrac{\beta^2}{\varepsilon_1} \\ \vartheta_3 = k\left\|\dot{\widehat{p}}\right\|, & \sigma_3 = 1, & S_{13} = k^2\,k_1(p^*,\widehat{p}(0)) \\ \varpi_1 = 2k\sqrt{r}\|e\|\,, & \zeta_1 = 2, & S_{21} = 4\,k^2\,\dfrac{\beta^2}{\varepsilon_1} \\ \varpi_2 = \overline{k}\left(\sqrt{r}\|e\|\right)^{\lambda}, & \zeta_2 = \dfrac{2}{\lambda}\,, & S_{22} = \overline{k}^2\,\dfrac{\beta^2}{\varepsilon_1} \\ \varpi_3 = k\left\|\dot{\widehat{p}}\right\|, & \zeta_3 = 1, & S_{23} = k^2\,k_1(p^*,\widehat{p}(0))\,. \end{array} \tag{529}$$

With property 3 of U in assumption RURS, it follows that there exists a constant Υ depending only on the initial conditions such that, for all t in $[0,T)$, we have:

$$\left\|\widehat{h}(t)\right\| \leq \Upsilon . \tag{530}$$

Then, with (523), this implies the existence of a constant α, depending only on the initial conditions, such that, for all t in $[0,T)$:

$$\begin{aligned} \|h(x(t),\widehat{p}(t))\| &\leq \left\|\widehat{h}(t)\right\| + \|e(t)\| && (531)\\ &\leq \alpha . && (532)\end{aligned}$$

We also know from (447) that $\widehat{p}(t) \in \mathcal{K}$, where $\mathcal{K}$ is the following compact subset of Π:

$$\mathcal{K} = \left\{ p \;\middle|\; \|p - p^*\| \leq \frac{\beta}{\sqrt{\varepsilon_2}} \right\} \bigcap \Pi_1 . \tag{533}$$

With assumption RBO, this proves the existence of a compact subset Γ of $\mathbb{R}^n$ such that:

$$x(t) \in \Gamma \quad \forall t \in [0,T) . \tag{534}$$

Finally, with the continuity of the functions r and h and the fact that:

$$\widehat{z_{\mathrm{f}}} = h(x,\widehat{p}) - z_{\mathrm{f}} , \tag{535}$$

we have established that the solution remains in a compact subset of the open set defined in (522). It follows by contradiction that $T = +\infty$ and that the time functions $x(t)$, $\widehat{p}(t)$, $u(t)$ and $\dot{\widehat{p}}(t)$ are bounded on $[0,+\infty)$. ¿From here, we conclude the proof exactly as in the proof of Proposition (440).

Case: Assumption RMC does not hold: The only difference with the previous case, is that we use

$$u = u_{\mathrm{n}}(x,\widehat{p}) \tag{536}$$

instead of (521), and the fact that U may depend on p. Hence, everything remains the same up to, but not including, equation (524). To get the equivalent of (524), we have to evaluate the time derivative of:

$$\widehat{U}(t) = U(h(x(t),\widehat{p}(t))) \tag{537}$$

along the solutions of (504). We get successively:

$$\begin{aligned} \dot{\widehat{U}} \leq& -c\,\widehat{U} + \frac{\partial U}{\partial p}(\widehat{h},\widehat{p})\,\dot{\widehat{p}} \\ &+ \frac{\partial U}{\partial h}(\widehat{h},\widehat{p}) \left[-\rho(x,\widehat{p},u)\,e + \left(f(\widehat{h}-e\,,\,\widehat{p}) - f(\widehat{h},p) \right) + \left(\frac{\partial h}{\partial p} + Z_{\mathrm{f}} \right) \dot{\widehat{p}} \right] \\ \leq& -c\,\widehat{U} \\ &+ d_9(\widehat{p}) d_{10}(\widehat{p}) \max\left\{1, \widehat{U}^{\lambda}\right\} (1 + \max\{1\,,\, U(h(x,\widehat{p}))^{\kappa}\})\, \|e\| \\ &+ d_8(\widehat{p},e) \max\left\{1, \widehat{U}^{\kappa+\lambda}\right\} \|e\| \\ &+ d_9(\widehat{p}) \max\left\{1\,,\, \widehat{U}^{\lambda}\right\} \left\|\dot{\widehat{p}}\right\| \\ &+ d_9(\widehat{p})\, d_{11}(\widehat{p}) \max\left\{1\,,\, \widehat{U}^{\lambda}\right\} \max\left\{1\,,\, U(h(x,\widehat{p}))^{1-\lambda}\right\} \left\|\dot{\widehat{p}}\right\| \\ &+ d_{12}(\widehat{p}) \max\{1\,,\, \widehat{U}\} \left\|\dot{\widehat{p}}\right\| . \end{aligned} \tag{538}$$

With Appendix C, we get similarly to (525):

$$\max\{1\,,\,U(h(x,\widehat{p}),\widehat{p})^{1-\lambda}\} \leq\ \delta\,\max\{1\,,\,\widehat{U}^{1-\lambda}\} + \delta\left[\max\{1\,,\,\widehat{U}^{\lambda(1-\lambda)}\}\left(\|e\|^{1-\lambda}\,+\,\|e\|\right)\right]. \quad (539)$$

And, since $0 \leq \lambda < 1$ implies:

$$\lambda\,(2-\lambda)\ <\ 1\,, \quad (540)$$

we have also:

$$\max\left\{1,\widehat{U}^{\lambda}\right\}\,\max\left\{1,\widehat{U}^{\lambda(1-\lambda)}\right\}\left(\|e\|^{1-\lambda}+\|e\|\right)\ \leq\ \left(1+\widehat{U}\right)\,\left(\|e\|^{1-\lambda}+\|e\|\right). \quad (541)$$

As in the previous case, this implies the existence of a constant k depending only on the initial conditions such that:

$$\dot{\widehat{U}} \leq -\left[c\,-\,k\left(\|e\|+\sqrt{r}\|e\|+\left(\sqrt{r}\|e\|\right)^{\lambda}+\left\|\dot{\widehat{p}}\right\|\right)\right]\widehat{U} + k\left(\|e\|+\sqrt{r}\|e\|\,+\,\left(\sqrt{r}\|e\|\right)^{\lambda}\,+\,\left\|\dot{\widehat{p}}\right\|\right). \quad (542)$$

We can now apply Lemma (583) in Appendix B with:

$$X\ =\ \widehat{U} \quad (543)$$

and:

$$\begin{array}{lll}
\vartheta_1\ =\ k\left(\|e\|+\sqrt{r}\|e\|\right), & \sigma_1\ =\ 2, & S_{11}\ =\ 2k^2\,\dfrac{\beta^2}{\varepsilon_1}\\
\vartheta_2\ =\ k\,\left(\sqrt{r}\|e\|\right)^{\lambda}, & \sigma_2\ =\ \dfrac{2}{\lambda}\,, & S_{12}\ =\ k^2\,\dfrac{\beta^2}{\varepsilon_1}\\
\vartheta_3\ =\ k\left\|\dot{\widehat{p}}\right\|, & \sigma_3\ =\ 1, & S_{13}\ =\ k^2\,k_1(p^*,\widehat{p}(0))\\
\varpi_1\ =\ k\left(\|e\|+\sqrt{r}\|e\|\right), & \zeta_1\ =\ 2, & S_{21}\ =\ 2k^2\,\dfrac{\beta^2}{\varepsilon_1}\\
\varpi_2\ =\ k\,\left(\sqrt{r}\|e\|\right)^{\lambda}, & \zeta_2\ =\ \dfrac{2}{\lambda}\,, & S_{22}\ =\ k^2\,\dfrac{\beta^2}{\varepsilon_1}\\
\varpi_3\ =\ k\left\|\dot{\widehat{p}}\right\|, & \zeta_3\ =\ 1, & S_{23}\ =\ k^2\,k_1(p^*,\widehat{p}(0))\,.
\end{array} \quad (544)$$

¿From here, we conclude as in the previous case. □

Again, in practice, to apply Proposition (512), we must be allowed to choose a particular value for the initial condition $\widehat{z_{\mathrm{f}}}(0)$, and a vanishing observer gain K ((263) is assumed). However, the main feature of this result together with Proposition (440) is that, when the observation function h is not related to V and assumption RMC is not satisfied, it provides solutions to the Adaptive Stabilization problem under different growth conditions.

Corollary (545)
Let, in equation (S_p)*, the functions* a *and* A *be* known *and affine in* u *and let* Π *be an open subset of* $\mathbb{R}^l$ *which satisfies assumption ICS. Assume there exist two* known *functions:*

$h\ :\mathbb{R}^n\times\Pi\ \to\ \mathbb{R}^n$ *of class* C^2 *which is a diffeomorphism for each* p*, and*
$u_{\mathrm{n}}:\mathbb{R}^n\times\Pi\ \to\ \mathbb{R}^m$ *of class* C^1 *,*

a positive real number κ *and a positive continuous function* d_{10}*, defined on* Π*, such that:*

1. *h satisfies assumption RCO,*
2. *for all (x,p) in $\mathbb{R}^n \times \Pi$, we have:*

$$\left\| \frac{\partial h}{\partial x}(x,p)\, A(x,u_{\mathrm{n}}(x,p)) \right\| \;\leq\; d_{10}(p)\, \max\{1\,, \|h(x,p)\|^{2\kappa}\}\,, \tag{546}$$

3. *by letting:*

$$\varphi \;=\; h(x,p)\,, \tag{547}$$

the time derivative of φ along the solutions of (S_p) with $u = u_{\mathrm{n}}$ satisfies:

$$\dot{\varphi} \;=\; C\,\varphi\,, \tag{548}$$

where C is an $n \times n$ matrix satisfying:

$$P\,C \,+\, C^{\mathrm{T}} P \;=\; -I\,, \tag{549}$$

with P a symmetric positive definite matrix.

Under these conditions, if

either *the function A does not depend on u and there exists a* known *function:*

$$v \,:\, \mathbb{R}^n \times \Pi \times \mathbb{R}^l \;\to\; \mathbb{R}^m \quad \textit{of class } C^1\,, \tag{550}$$

such that, for all (x,p,∂) in $\mathbb{R}^n \times \Pi \times \mathbb{R}^l$, we have:

$$\frac{\partial h}{\partial p}(x,p)\,\partial \,+\, \frac{\partial h}{\partial x}(x,p)\, v \odot b(x) \;=\; 0\,, \tag{551}$$

or *there exists a positive continuous function d_{11}, defined on Π such that, for all (x,p) in $\mathbb{R}^n \times \Pi$:*

$$\left\| \frac{\partial h}{\partial p}(x,p) \right\| \;\leq\; d_{11}(p)\, \max\{1\,, \|h(x,p)\|^2\}\,, \tag{552}$$

then the Adaptive Stabilization problem is solved by the dynamic controller consisting of (259), (261), (262, (516), and:

$$u \;=\; u_{\mathrm{n}}(x,\widehat{p}) \,+\, v\left(x, \widehat{p}, \widehat{q}, \dot{\widehat{p}}\right) \tag{553}$$

if v exists, or:

$$u \;=\; u_{\mathrm{n}}(x,\widehat{p}) \tag{554}$$

if not.

Proof. This proof follows exactly the same lines as the proof of Corollary (485). In particular, the above growth conditions are nothing but GC2 with:

$$\lambda \;=\; \frac{1}{2}\,, \quad U(h) \;=\; h^{\mathrm{T}} P\, h \qquad \text{and} \qquad f(h\,,\,p) \;=\; C\,h\,. \tag{555}$$

□

This Corollary (545) should be compared with the result of Nam and Arapostathis in [16]. In the same context of adaptive feedback linearization, they propose the same dynamic controller except that:

1. the Matching Condition (551) is not assumed,
2. they do not restrict the choice of $\widehat{z_{\mathrm{f}}}(0)$,

3. the observer gain is not allowed to go to zero, namely, (263) does not hold,
4. ρ is kept constant,
5. finally r is given by:

$$r = \left(1 + \left\|\frac{\partial h}{\partial x}(x,p)\, A(x, u_{\mathrm{n}}(x,p))\right\|^2\right)^{-1}, \tag{556}$$

i.e., they impose $\kappa = 1$.

As a consequence, Nam and Arapostathis get a solution to the Adaptive Stabilization problem under more restrictive Growth Conditions. Namely, instead of (546), they assume:

$$\left\|\frac{\partial h}{\partial x}(x,p)\, A(x, u_{\mathrm{n}}(x,p))\right\| \leq d_{10}(p)\ \max\{1\,, \|h(x,p)\|\}\,, \tag{557}$$

and, instead of (552), they have:

$$\left\|\frac{\partial h}{\partial p}(x,p)\right\| \leq d_{11}(p)\ \max\{1\,, \|h(x,p)\|\}\,. \tag{558}$$

6 Conclusion

In this paper, we have given a unified and generalizing overview of most (not all!) of the presently proposed approaches to stabilize an equilibrium point of a nonlinear system described by a differential equation containing unknown parameters. Table 3 summarizes our results.

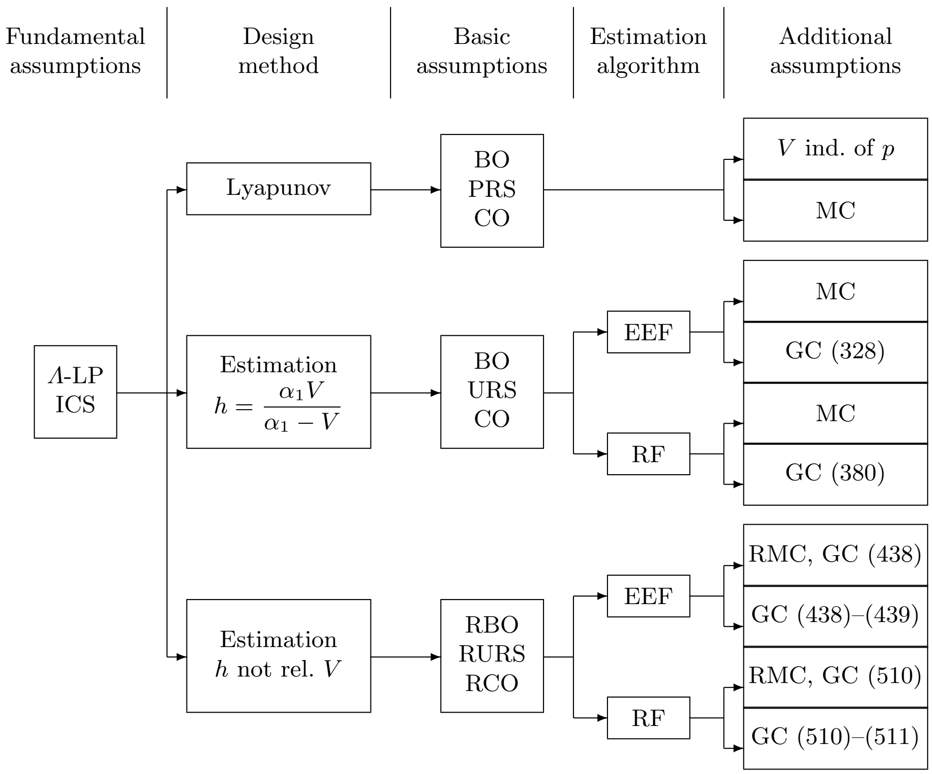
A key assumption is the fact that the right-hand side of the differential equation describing the system depends linearly on these unknown parameters or at least on those actually needed for the control (see Example (49)). To meet this assumption, called Λ-Linear Parameterization Λ-LP, we have mentioned the fact that it may be useful not to work with the a priori given coordinates and parameterization: a parameter-dependent change of coordinates and a reparameterization, i.e., a transformation of the parameters, are allowed (see [14] and [29]). Also, Middleton and Goodwin [15], Pomet and Praly [22] and Slotine and Li [30], for example, have shown that the proposed results extend in some cases to a more general case called Implicit Linear Parameterization in [19]. Finally, Mareels, Penfold and Evans [13] have shown that it is also possible to follow a non parametric approach to solve the problem of stabilizing an equilibrium point of a system whose dynamics are not completely known.

Another important assumption is assumption PRS and its more restrictive versions URS and RURS. It guarantees not only that the system is stabilizable, but also that a parameter-dependent control law for this stabilization is known. It follows that the only problem addressed here is: how can we use this control law when the parameters are unknown, i.e., how can we make this control law adaptive?

Table 3 shows the ten routes we have studied for designing and analyzing adaptive controllers of nonlinear systems. It shows the interplay between structural assumptions, control design techniques and estimation algorithms. In particular, it emphasizes the fact that any stabilizing control law cannot necessarily be made adaptive. It has to give the closed-loop system properties which differ depending on which control design and estimation technique is used. However, a general very desirable property is the fact that the so-called Matching Condition (MC) can be satisfied.

Implicit in all what has been presented was the assumption that the state is completely measured. It follows that the problem we are addressing is a very particular case of the Error Feedback Regulator problem stated by Isidori [6, Sect. 7.2]. But thanks to this particularity, we have solved the problem under less restrictive assumptions. It is also worth mentioning that the results established by Kanellakopoulos, Kokotovic and Middleton [10] lead us to expect that relaxation of the assumptions is also possible in some cases when the state is not completely measured.

Table 3. Summary of results



BO : Boundedness Observability
EEF : Equation Error Filtering
GC : Growth Condition
MC : Matching Condition
R : Refined
CO : Convergence Observability
Λ-LP : Λ-Linear Parameterization
ICS : Imbedded Convex Sets
PRS : Pointwise Reduced-order Stabilizability
RF : Regressor Filtering
URS : Uniform Reduced-order Stabilizability

Appendices

A: Proof of Lemma (103)

In this proof, we denote by S the following open subset of $\Pi \times \mathbb{R}^l$:

$$S = \left\{ (p, y) \,\middle|\, \mathcal{P}(p) > 0 \ \text{ and } \ \frac{\partial \mathcal{P}}{\partial p}(p)\, y > 0 \right\} . \tag{559}$$

$\mathrm{Proj}(M, p, y)$ differs from y if and only if (p, y) belongs to S.

Point 1: We make the following preliminary remark:
Since $0 \leq \mathcal{P}(p)$ and $M \in \mathcal{M}$ imply:

$$\frac{\partial \mathcal{P}}{\partial p}(p)\, M\, \frac{\partial \mathcal{P}}{\partial p}(p)^{\mathrm{T}} > 0 , \tag{560}$$

and $\mathcal{P}$ is a twice continuously differentiable function,

1. the function $\mathrm{Proj}(M, p, y)$ is continuously differentiable in the set $\mathcal{M} \times S$,
2. $\mathrm{Proj}(M, p, y)$ tends to y as $\mathcal{P}(p)$ or $\frac{\partial \mathcal{P}}{\partial p}(p)\, y$ tends to 0,
3. for any compact subset $\mathcal{C}$ of

$$\mathcal{M} \times \left\{ (p, y) \,\middle|\, \mathcal{P}(p) \geq 0 \ \text{ and } \ \frac{\partial \mathcal{P}}{\partial p}(p)\, y \geq 0 \right\} ,$$

there exists a constant $k_{\mathcal{C}}$ bounding the Jacobian matrix:

$$\|\nabla \, \mathrm{Proj}(M, p, y)\| \leq k_{\mathcal{C}} \qquad \forall\, (M, p, y) \in \mathcal{C} . \tag{561}$$

Now, let (M_1, p_1, y_1) and (M_0, p_0, y_0) be two points such that, for any α in $[0, 1]$, the point $(M_\alpha, p_\alpha, y_\alpha)$ is in the set $\mathcal{M} \times \Pi \times \mathbb{R}^l$, with:

$$\begin{aligned} M_\alpha &= \alpha\, M_1 + (1 - \alpha)\, M_0 \\ p_\alpha &= \alpha\, p_1 + (1 - \alpha)\, p_0 \\ y_\alpha &= \alpha\, y_1 + (1 - \alpha)\, y_0 . \end{aligned} \tag{562}$$

Four cases must be distinguished:

Case 1: (p_1, y_1) and (p_0, y_0) are not in S. Then, we have trivially:

$$\|\mathrm{Proj}(M_1, p_1, y_1) - \mathrm{Proj}(M_0, p_0, y_0)\| = \|y_1 - y_0\| . \tag{563}$$

Case 2: For all α in $[0, 1]$, (p_α, y_α) lies in S. Then, from the above preliminary remark and the Mean Value Theorem, we get:

$$\|\mathrm{Proj}(M_1, p_1, y_1) - \mathrm{Proj}(M_0, p_0, y_0)\| \leq k\, [\|M_1 - M_0\| + \|p_1 - p_0\| + \|y_1 - y_0\|] \tag{564}$$

with the constant k given by (561).

Case 3: Say (p_0, y_0) belongs to S but (p_1, y_1) does not. Then, we define α^* by:

$$\alpha^* = \min_{\substack{0 \leq \alpha \leq 1 \\ (p_\alpha, y_\alpha) \notin S}} \alpha . \tag{565}$$

Since S is open, all the points of the segment $[\,(M_0,p_0,y_0)\,,\,(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*})\,)$ have their (p,y) component lying in S. But $(p_{\alpha^*},y_{\alpha^*})$ is not in S. With (561) and (562), this implies:

$$\begin{aligned}&\|\mathrm{Proj}(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*})-\mathrm{Proj}(M_0,p_0,y_0)\|\\&\qquad\le\;k\,[\|M_1-M_0\|+\|p_1-p_0\|+\|y_1-y_0\|]\,.\end{aligned}\tag{566}$$

We also have:

$$\|\mathrm{Proj}(M_1,p_1,y_1)-\mathrm{Proj}\,(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*})\|\;=\;\|y_1-y_{\alpha^*}\|\;\le\;\|y_1-y_0\|\;.\tag{567}$$

This yields:

$$\begin{aligned}&\|\mathrm{Proj}(M_1,p_1,y_1)-\mathrm{Proj}(M_0,p_0,y_0)\|\\&\qquad\le\;(1+k)\,[\|M_1-M_0\|+\|p_1-p_0\|+\|y_1-y_0\|]\,.\end{aligned}\tag{568}$$

Case 4: Finally, when both (p_1,y_1) and (p_0,y_0) belong to S, but there are some α in $(0,1)$ for which (p_α,y_α) is not in S, we define α^* as above and let:

$$\beta^*\;=\;\max_{\substack{0\le\beta\le1\\(p_\beta,y_\beta)\notin S}}\beta\,.\tag{569}$$

Then all the points of the segments $[(M_0,p_0,y_0)\,,\,(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*}))$ and $((M_{\beta^*},p_{\beta^*},y_{\beta^*})\,,\,(M_1,p_1,y_1)]$ have their (p,y) component in S. But the points $(p_{\alpha^*},y_{\alpha^*})$ and $(p_{\beta^*},y_{\beta^*})$ are not in S. With (561), we get:

$$\begin{aligned}&\|\mathrm{Proj}(M_1,p_1,y_1)-\mathrm{Proj}(M_{\beta^*},p_{\beta^*},y_{\beta^*})\|\\&+\|\mathrm{Proj}(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*})-\mathrm{Proj}(M_0,p_0,y_0)\|\\&\qquad\le\;2k\,[\|M_1-M_2\|+\|p_1-p_2\|+\|y_1-y_2\|]\,.\end{aligned}\tag{570}$$

The conclusion follows, since we have trivially:

$$\mathrm{Proj}(M_{\beta^*},p_{\beta^*},y_{\beta^*})-\mathrm{Proj}(M_{\alpha^*},p_{\alpha^*},y_{\alpha^*})=y_{\beta^*}-y_{\alpha^*}\;.\tag{571}$$

Point 2: For (p,y) not in S the inequality of point 2 is trivial. If (p,y) is in S, a direct computation gives:

$$\mathrm{Proj}(M,p,y)^{\mathrm{T}}M^{-1}\mathrm{Proj}(M,p,y)=y^{\mathrm{T}}M^{-1}y-\frac{\mathcal{P}(p)\;(2-\mathcal{P}(p))\;\left(\frac{\partial\mathcal{P}}{\partial p}(p)y\right)^2}{\frac{\partial\mathcal{P}}{\partial p}(p)M\,\frac{\partial\mathcal{P}}{\partial p}(p)^{\mathrm{T}}}\,.\tag{572}$$

The conclusion follows since, by definition, for all (p,y) in S with p in Π_1, we have:

$$\mathcal{P}(p)\;(2-\mathcal{P}(p))\;>\;0\,.\tag{573}$$

Point 3: For any p satisfying $\mathcal{P}(p)\;\ge\;0$, let q be the orthogonal projection of p^* on the hyperplane through p and orthogonal to $\frac{\partial\mathcal{P}}{\partial p}(p)$, i.e.,

$$q\;=\;p^*\;-\;\frac{\frac{\partial\mathcal{P}}{\partial p}(p)(p-p^*)}{\left\|\frac{\partial\mathcal{P}}{\partial p}(p)\right\|^2}\,\frac{\partial\mathcal{P}}{\partial p}(p)^{\mathrm{T}}\;.\tag{574}$$

Since $\mathcal{P}$ is a convex function we have:

$$\mathcal{P}(q) \geq \mathcal{P}(p) + \frac{\partial \mathcal{P}}{\partial p}(p)\,(q-p) \tag{575}$$
$$\geq \mathcal{P}(p) \geq 0\,. \tag{576}$$

It follows that q is not an interior point of the set Π_0 and, therefore, $\|q-p^*\|$ is larger or equal to D^*, the distance from p^* to the boundary of the closed set $\Pi_0 = \{p \mid \mathcal{P}(p) = 0\}$, i.e., we have:

$$\frac{\left(\frac{\partial \mathcal{P}}{\partial p}(p)(p-p^*)\right)^2}{\left\|\frac{\partial \mathcal{P}}{\partial p}(p)\right\|^2} \geq D^{*2}\,. \tag{577}$$

The conclusion follows from the fact that, p^* being an interior point of the set Π_0 (see (95) and (93)), the Basic Separation Hahn-Banach Theorem [4, Theorem V.1.12] implies:

$$\frac{\partial \mathcal{P}}{\partial p}(p)(p-p^*) > 0 \quad \forall\, p : \mathcal{P}(p) \geq 0\,. \tag{578}$$

Point 4: Again, point 4 is clear for all (p,y) not in S. And, for (p,y) in S, we get, using point 3:

$$\begin{aligned}
&(p-p^*)^{\mathrm{T}} M^{-1}\mathrm{Proj}(M,p,y) \\
&\quad = (p-p^*)^{\mathrm{T}} M^{-1} y - \frac{\mathcal{P}(p)\left(\frac{\partial \mathcal{P}}{\partial p}(p)y\right)\left(\frac{\partial \mathcal{P}}{\partial p}(p)(p-p^*)\right)}{\frac{\partial \mathcal{P}}{\partial p}(p) M \frac{\partial \mathcal{P}}{\partial p}(p)^{\mathrm{T}}} \\
&\quad \leq (p-p^*)^{\mathrm{T}} M^{-1} y\,.
\end{aligned} \tag{579}$$

Point 5: Let us compute the time derivative of $\mathcal{P}(p(t))$ along a solution of (104). We get:

$$\begin{aligned}
\dot{\overline{\mathcal{P}(\widehat{p}(t))}} &= \tfrac{\partial \mathcal{P}}{\partial p}(\widehat{p}(t))y(t) && \text{if } \mathcal{P}(\widehat{p}(t)) \leq 0 \text{ or } \tfrac{\partial \mathcal{P}}{\partial p}(\widehat{p}(t))\,y(t) \leq 0 \\
&= \tfrac{\partial \mathcal{P}}{\partial p}(\widehat{p}(t))y(t)\,(1-\mathcal{P}(\widehat{p}(t))) && \text{if } \mathcal{P}(\widehat{p}(t)) > 0 \text{ and } \tfrac{\partial \mathcal{P}}{\partial p}(\widehat{p}(t))\,y(t) > 0\,.
\end{aligned} \tag{580}$$

Therefore, we have:

$$\dot{\overline{\mathcal{P}(\widehat{p}(t))}} \leq 0 \qquad \text{if} \qquad \mathcal{P}(\widehat{p}(t)) \geq 1\,. \tag{581}$$

Since the initial condition satisfies:

$$\mathcal{P}(\widehat{p}(0)) \leq 1\,, \tag{582}$$

a continuity argument proves that the same holds for all t where $\widehat{p}(t)$ is defined.

□

B: A Useful Technical Lemma

Lemma: (see also [3, Theorem IV.1.9]) **(583)**
Let X be a C^1 time function defined on $[0\,,T)$ $(0<T\leq+\infty)$, satisfying:

$$\dot{X} \;\leq\; -c\,X \;+\; \sum_i \vartheta_i(t)\,X(t) \;+\; \sum_j \varpi_j(t)\,, \tag{584}$$

where c is a strictly positive constant, $\sum_i$ and $\sum_j$ are finite sums and ϑ_i, and ϖ_j are positive time functions satisfying:

$$\int_0^T {\vartheta_i}^{\sigma_i} \;\leq\; S_{1i} \qquad \text{and} \qquad \int_0^T {\varpi_j}^{\zeta_j} \;\leq\; S_{2j}\,, \tag{585}$$

where $\sigma_i\geq 1$ and $\zeta_j\geq 1$. Under this assumption, $X(t)$ is bounded from above on $[0,T)$ and, precisely:

$$X(t) \;\leq\; K_1\,X(0)+K_2 \qquad \forall\,t\,\in\,[0,T)\,, \tag{586}$$

with K_1 and K_2 depending only on σ_i, ζ_j, S_{1i} and S_{2j}.
Moreover, if T is infinite then:

$$\limsup_{t\to\infty}\, X(t) \;\leq\; 0\,. \tag{587}$$

Proof. This is a straightforward consequence of a known result on differential inequalities. ¿From (584), one derives (see [5, Theorem I.6.1]):

$$X(t) \;\leq\; X(0)\,e^{-ct+\displaystyle\int_0^t \sum_i \vartheta_i} + \int_0^t e^{\left[-c(t-\tau)+\displaystyle\int_\tau^t \sum_i \vartheta_i\right]} \sum_j \varpi_j(\tau)\,\mathrm{d}\tau\,. \tag{588}$$

Let $\theta_i>1$ and $\eta_j>1$ be defined by:

$$\frac{1}{\sigma_i}+\frac{1}{\theta_i} \;=\; 1 \qquad \text{and} \qquad \frac{1}{\zeta_j}+\frac{1}{\eta_j} \;=\; 1\,. \tag{589}$$

Inequalities (585) and the Hölder Inequality yield, for any positive t and τ, $t\geq\tau$:

$$\int_\tau^t \vartheta_i \leq (t-\tau)^{\frac{1}{\theta_i}} \left(\int_\tau^t {\vartheta_i}^{\sigma_i}\right)^{\frac{1}{\sigma_i}} \tag{590}$$

$$\leq (t-\tau)^{\frac{1}{\theta_i}}\; {S_{1i}}^{\frac{1}{\sigma_i}}\,. \tag{591}$$

Similarly, we get:

$$\begin{aligned}
\int_0^t e^{-\frac{c}{2}(t-\tau)}\varpi_j(\tau)\mathrm{d}\tau &\leq \left(\frac{2}{c}\right)^{\frac{1}{\eta_j}} \left(\int_0^t e^{-\frac{c}{2}(t-\tau)}\varpi_j(\tau)^{\zeta_j}\mathrm{d}\tau\right)^{\frac{1}{\zeta_j}} \\
&\leq \left(\frac{2}{c}\right)^{\frac{1}{\eta_j}} \left(e^{-\frac{c}{2}\frac{t}{2}}\int_0^{\frac{t}{2}} \varpi_j(\tau)^{\zeta_j}\,\mathrm{d}\tau \,+\, \int_{\frac{t}{2}}^t \varpi_j(\tau)^{\zeta_j}\,\mathrm{d}\tau\right)^{\frac{1}{\zeta_j}} \\
&\leq \left(\frac{2}{c}\right)^{\frac{1}{\eta_j}} \left(e^{-\frac{c}{2}\frac{t}{2}}S_{2j} \,+\, \int_{\frac{t}{2}}^t \varpi_j(\tau)^{\zeta_j}\,\mathrm{d}\tau\right)^{\frac{1}{\zeta_j}}\,.
\end{aligned} \tag{592}$$

Then, let us note that the function:

$$f(x) \,=\, -\frac{c}{2}\,x \,+\, \sum_i \, S_{1i}{}^{\frac{1}{\sigma_i}}\, x^{\frac{1}{\theta_i}} \tag{593}$$

is well-defined and continuous on $[0,+\infty)$, with:

$$f(0) \,=\, 0 \qquad \text{and} \qquad f(+\infty) \,=\, -\infty\,. \tag{594}$$

This implies the existence of a constant K_1 depending only on S_{1i} and σ_i such that, for all $0 \leq \tau \leq t$:

$$\exp\left(-\frac{c}{2}\,(t-\tau) + \sum_i (t-\tau)^{\frac{1}{\theta_i}}\, S_{1i}{}^{\frac{1}{\sigma_i}} \right) \,\leq\, K_1\,. \tag{595}$$

With this bound, (588) and (592) we have established:

$$X(t) \,\leq\, K_1 \left[e^{-\frac{c}{2}t} X(0) + \sum_j \left(\frac{2}{c}\right)^{\frac{1}{\eta_j}} \left(e^{-\frac{c}{2}\frac{t}{2}} S_{2j} \,+\, \int_{\frac{t}{2}}^{t} \varpi_j(\tau)^{\zeta_j}\,\mathrm{d}\tau \right)^{\frac{1}{\zeta_j}} \right]. \tag{596}$$

The proof is completed by noticing that (585) implies that, if $T = +\infty$, we have:

$$\lim_{t\to+\infty} \int_{\frac{t}{2}}^{t} \varpi_j(\tau)^{\zeta_j}\,\mathrm{d}\tau \,=\, 0\,. \tag{597}$$

□

C: An Inequality

Lemma: **(598)**
Let $U \,:\, \mathbb{R}^k \,\to\, \mathbb{R}_+$ be a C^1 function such that:

$$\left\| \frac{\partial U}{\partial h}(h) \right\| \,\leq\, d\, \max\left\{1, U(h)^\lambda\right\} \qquad \forall\, h \,\in\, \mathbb{R}^k\,, \tag{599}$$

with $0 \leq \lambda < 1$ and d a positive constant. For any positive real number γ, there exists a constant δ such that, for all (h,e) in $\mathbb{R}^k \times \mathbb{R}^k$, we have:

$$\begin{aligned} \max\{1\,,\,U(h-e)^\gamma\} \leq \delta\, \max\{1\,,\,U(h)^\gamma\} & \\ +\delta \left[\max\left\{1\,,\,U(h)^{\lambda\gamma}\right\} \left(\|e\|^\gamma + \|e\|^{\frac{\gamma}{1-\lambda}} \right) \right]. & \end{aligned} \tag{600}$$

Proof. Let us define a function W as follows:

$$W(h) \,=\, \max\left\{1\,,\,U(h)^{1-\lambda}\right\}\,. \tag{601}$$

This function is of class C^1 in the open set $\{h \mid U(h) > 1\}$ with:

$$\left\| \frac{\partial W}{\partial h}(h) \right\| \,\leq\, d\,(1-\lambda)\,. \tag{602}$$

Then, for all (h,e) in $\mathbb{R}^k \times \mathbb{R}^k$, we have:

$$W(h-e) - W(h) \,\leq\, d\,(1-\lambda)\,\|e\|\,. \tag{603}$$

This is proved by breaking the segment $[h-e\,,\,h]$ into pieces depending on whether U is larger than 1 or not and by noting that $W(h-e)-W(h)$ is equal to the sum of the variations of W on the two extreme pieces only (see the proof of Point 1 in Appendix A). This yields:

$$\max\{1, U(h-e)^\gamma\} - \max\{1, U(h)^\gamma\} \;\leq\; (W(h) + d\,(1-\lambda)\,\|e\|)^{\frac{\gamma}{1-\lambda}} - W(h)^{\frac{\gamma}{1-\lambda}}\,. \tag{604}$$

Let us now note that, for all $\tau \geq 0$, the function:

$$f(x) \;=\; \frac{(1+x)^\tau - x^\tau}{x^\tau + x^{\lambda\tau} + 1} \tag{605}$$

is positive, well-defined and continuous on $[0, +\infty)$, with:

$$f(0) \;=\; 1 \qquad \text{and} \qquad f(+\infty) \;=\; 0\,. \tag{606}$$

This implies the existence of a constant K depending only on γ and λ such that:

$$\begin{aligned} &\max\{1\,,\,U(h-e)^\gamma\} - \max\{1\,,\,U(h)^\gamma\} \\ &\leq\; K\left[W(h)^{\frac{\gamma}{1-\lambda}} + W(h)^{\frac{\lambda\gamma}{1-\lambda}}\,(d(1-\lambda)\,\|e\|)^\gamma + (d(1-\lambda)\,\|e\|)^{\frac{\gamma}{1-\lambda}}\right]. \end{aligned} \tag{607}$$

The conclusion follows readily. □

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